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Semi-Fredholm Operators and Periodic Solutions for Linear Functional Differential Equations

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We deal with the inhomogeneous linear periodic equation with infinite delay of the

on the existence and the uniqueness of periodic solutions for the equation in the general phase space $\mathscr{B}$ and in the concrete phase space $\mathscr{B} = UC_g$. The key of our approach is the employment of the perturbation theory of semi-Fredholm operators to show that the period map satisfies the condition of the fixed point theorem by Chow and Hale (*Funkcial. Ekvac.* **17** (1974), 31–38).

1. INTRODUCTION

Let R be a real line, and E a Banach space with a norm $|\cdot|$. If $x: (-\infty, a) \to E$, then a function $x_t: (-\infty, 0] \to E$, $t \in (-\infty, a)$, is defined by $x_t(\theta) = x(t+\theta)$, $\theta \in (-\infty, 0]$. We deal with the linear functional differential equation with infinite delay in the Banach space E:

$$\text{(L)} \qquad \frac{dx(t)}{dt} = Ax(t) + B(t, x_t) + F(t).$$

Let $\mathscr{B}$ be a Banach space, consisting of functions $\psi: (-\infty, 0] \to E$, which satisfies some axioms demonstrated in Section 2. We assume that Eq. (L) always satisfies the following hypothesis (H):

(H-1) $A: \mathscr{D}(A) \subset E \to E$ is the infinitesimal generator of a C_0-semigroup $T(t)$ on E;

(H-2) $B: R \times \mathscr{B} \to E$ is continuous and $B(t, \cdot): \mathscr{B} \to E$ is linear;

(H-3) $F: R \to E$ is continuous.

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If $B(t, \psi)$ and $F(t)$ in Eq. (L) are periodic functions with a period $\omega > 0$, we denote Eq. (L) by Eq. (P_ωL). If $F \equiv 0$, we denote Eq. (L) and Eq. (P_ωL) by Eq. (L_0) and Eq. ($\mathrm{P}_\omega \mathrm{L}_0$), respectively.

Chow and Hale [2] obtained the following two fixed point theorems for a linear affine map on a Banach space. Let X and Y be Banach spaces, $B(X, Y)$ the set of bounded linear operators on X into Y and $\|L\|$ the operator norm of $L \in B(X, Y)$. Let $T: X \to X$ be a linear affine map $Tx = Lx + z$, $x \in X$, where $L \in B(x) := B(X, X)$ and $z \in X$ is fixed.

THEOREM A. *If the range $R(I - L)$ is closed and if there is an $x_0 \in X$ such that $\{x_0, Tx_0, T^2x_0, ...\}$ is bounded in X, then T has a fixed point in X.*

THEOREM B. *If there is an $x_0 \in X$ such that $\{x_0, Tx_0, T^2x_0, ...\}$ is relatively compact in X, then T has a fixed point in X.*

The proofs of Theorems A and B are based on the Hahn-Banach theorem and Schauder's fixed point theorem, respectively. They applied Theorem A to Eq. (P_ωL) with finite delay for the case where $A = 0$ and $E = R^n$ to show that, if there exists a bounded solution, then there exists a periodic solution of period ω. We consider the similar property for Eq. (P_ωL) in general:

(*BP*). If Eq. (P_ωL) has a solution which is bounded in $\mathscr{B}$, there exists a periodic solution of period ω of Eq. (P_ωL).

This property does not hold without any assumption for the equation with infinite delay in the Banach space. The main purpose of this paper is to find when our equation Eq. (P_ωL) has Property (BP) with respect to the mild solutions, see Section 2.

The essential part of the proof in [2] is the following one: if L is an α-contraction, then the range $R(I - L)$ is closed [1]. But we emphasize that for Eq. (P_ωL), L is not an α-contraction in general.

More recently, using Theorem B, Hino, Murakami and Yoshizawa obtained the following result: if the phase space $\mathscr{B}$ is a fading memory space and if the C_0-semigroup $T(t)$ is compact for $t > 0$, then Eq. (P_ωL) has an ω-periodic, mild solution as long as it has a mild solution which is E-bounded for $t \geqslant 0$. The compactness of the C_0-semigroup $T(t)$ for $t > 0$ on E plays an essential role in its proof, cf. [7]. In such a direction, Li, Lim and Li [9] have also considered the existence of periodic solutions of Eq. (P_ωL) with advanced and delay for the case where $A = 0$ and $E = R^n$. However, Theorem B cannot apply even for the case where $B(t, \cdot)$ is a compact operator for each $t \in R$, or C_0-semigroup $T(t)$ is compact only for $t \geqslant t_0$, where t_0 is a positive constant.

In this paper, we deal with Property (BP) for Eq. (P_ωL) by using Theorem A. Our manner is useful for the above exceptional cases, too. In general, it is difficult to see the closedness of the range $R(I-L)$ in Theorem A, because L is not an α-contraction as mentioned above. For such a reason, it seems that Theorem A has little effect in applications for a long time. Nevertheless, we will show that Theorem A is indeed powerful for Eq. (P_ωL), in Banach spaces. To do this, we employ the theory of semi-Fredholm operators. Suppose that $F \in B(X, Y)$. If $R(F)$ is closed, and if the null space $N(F)$ is of finite dimension, then F is called a semi-Fredholm operator. $\Phi_+(X, Y)$ denotes the set of all semi-Fredholm operators, and $\Phi_+(X) := \Phi_+(X, X)$. Denote by $\mathscr{F}_T$ the set of the fixed points of the affine map T. If we find some $y \in \mathscr{F}_T$, then $\mathscr{F}_T = y + N(I-L)$: it is an affine space. Define $\dim \mathscr{F}_T := \dim N(I-L)$. If $I - L \in \Phi_+(X)$, then Theorem A is refined as follows.

THEOREM 1.1. *Let $Tx = Lx + z$ be a linear affine map on a Banach space X as in the avobe. If $I - L \in \Phi_+(X)$ and there exists an $x_0 \in X$ such that $\{x_0, Tx_0, T^2x_0, ...\}$ is bounded, then $\mathscr{F}_T \neq \emptyset$ and* dim *$\mathscr{F}_T$ is finite.*

The semi-Fredholm operators have advantages in the perturbation. In fact, if $F \in \Phi_+(X, Y)$, then (i) $F + K \in \Phi_+(X, Y)$ for any compact linear operator $K: X \to Y$; (ii) there exists a positive constant η such that, if $S \in B(X, Y)$ satisfies $\|S\| < \eta$, then $F + S \in \Phi_+(X, Y)$ and $\dim N(F + S) \leqslant \dim N(F)$, see [4, 15].

Now we will explain our manners in more detail. If an initial condition $x_0 = \phi \in \mathscr{B}$ is given, then Eq. ($\mathrm{P}_\omega \mathrm{L}_0$) has a unique mild solution $x(t, \phi)$, see Section 2. The solution operator $U(t, 0)$ of Eq. ($\mathrm{P}_\omega \mathrm{L}_0$) defined by $U(t, 0)\, \phi = x_t(\phi)$, $t \geqslant 0$, is decomposed as $U(t, 0)\, \phi = \hat{T}(t)\, \phi + K(t, 0)\, \phi$, where

$$[\hat{T}(t)\, \phi](\theta) = \begin{cases} T(t+\theta)\, \phi(0) & \text{for} \quad t + \theta \geqslant 0, \\ \phi(t+\theta) & \text{for} \quad t + \theta \leqslant 0. \end{cases}$$

The existence of periodic solutions of Eq. (P_ωL) is equivalent to the existence of fixed points of the linear affine map in the space $\mathscr{B}$ defined by $\phi \mapsto U(\omega, 0)\, \phi + \psi$, where $\psi \in \mathscr{B}$ is a fixed function determined by a special solution of Eq. (P_ωL). It follows from Theorem 1.1 that if $I - U(\omega, 0) \in \Phi_+(\mathscr{B})$, then Property (BP) holds for Eq. (P_ωL). Furthermore, the perturbation theory of semi-Fredholm operators implies the following result. The proof is almost trivial.

THEOREM 1.2. *Suppose that $I - \hat{T}(\omega) \in \Phi_+(\mathscr{B})$. Then*

(1) *if $K(\omega, 0)$ is a compact operator, Eq. ($P_\omega L$) has Property (BP);*

(2) *there exists an* $\eta > 0$ *such that if* $\|K(\omega, 0)\| < \eta$, *Eq.* $(P_\omega L)$ *has Property* (*BP*) *and the set of periodic solutions makes a linear affine space with dimension* $\leqslant \dim N(I - \hat{T}(\omega))$.

Thus we have two problems. The first one is to check that $I - \hat{T}(\omega) \in \Phi_+(\mathscr{B})$. To do so, we will use the α-measure of noncompactness in the general phase space $\mathscr{B}$. In particular, we prove that one is a normal point of $\hat{T}(\omega)$ by showing that the essential spectral radius of $\hat{T}(\omega)$ is less than one, see Sections 2 and 4. On the other hand, if we take the concrete phase space $\mathscr{B} = UC_g$, and if it is a uniform fading memory space, we show directly that, if $I - T(\omega) \in \Phi_+(E)$, then $I - \hat{T}(\omega) \in \Phi_+(\mathscr{B})$, see Section 6. *In this case, one may be an accumulation point of the spectrum of* $\hat{T}(\omega)$; *that is, it may belong to its essential spectrum.* Anyway, the phase space $\mathscr{B}$ and $T(t)$ quite concern the first problem. The second problem is to investigate properties of the perturbation term $K(\omega, 0)$. Besides the phase space $\mathscr{B}$, the operators $T(t)$ and $B(t, \cdot)$ in Eq. $(\mathrm{P}_\omega \mathrm{L})$ concern the operator $K(\omega, 0)$. If they have some kind of compact property, $K(\omega, 0)$ also becomes compact. If they have not such a property, we employ the condition about $\|K(\omega, 0)\|$, which is estimated in terms of $\|T(t)\|$, $\|B(t, \cdot)\|$, see Section 3.

Combining these results, we will show several conditions, in Section 5 and 7, that Eq. $(\mathrm{P}_\omega \mathrm{L})$ has Property (BP). We emphasize that, even if $T(t)$ and $B(t, \cdot)$ have not the property of compactness, Eq. $(\mathrm{P}_\omega \mathrm{L})$ has Property (BP) provided that $\sup_{0 \leqslant t \leqslant \omega} \|B(t, \cdot)\|$ is sufficiently small. Finally, in Section 8 we will show the unique existence of a periodic solution by applying the result about the dimension of the set of periodic solutions.

2. THE PHASE SPACE $\mathscr{B}$ AND ESTIMATES OF SOLUTIONS

At first, we present the phase space $\mathscr{B}$ and a property of α-measure of noncompactness in relation with $\mathscr{B}$. Let $\mathscr{B}$ be a normed linear space consisting of some functions mapping $(-\infty, 0]$ into E; the norm in $\mathscr{B}$ is denoted by $|\cdot|_{\mathscr{B}}$. Throughout this paper we assume that $\mathscr{B}$ satisfies the following axioms.

(B-1). If a function $x: (-\infty, \sigma + a) \to E$ is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathscr{B}$, then

(i) $x_t \in \mathscr{B}$ for all $t \in [\sigma, \sigma + a)$ and x_t is continuous in $t \in [\sigma, \sigma + a)$;

(ii) $H^{-1} |x(t)| \leqslant |x_t|_{\mathscr{B}} \leqslant K(t - \sigma) \sup\{|x(s)| : \sigma \leqslant s \leqslant t\} + M(t - \sigma)|x_\sigma|_{\mathscr{B}}$ for all $t \in [\sigma, \sigma + a)$, where $H > 0$ is constant, $K: [0, \infty) \to [0, \infty)$ is continuous, $M: [0, \infty) \to [0, \infty)$ is measurable, locally bounded and they are independent of x.

(B-2) The space $\mathscr{B}$ is complete.

See [6] for several examples of $\mathscr{B}$. Let BC be the set of bounded, continuous functions mapping $(-\infty, 0]$ into E, and C_{00} its subset consisting of functions with compact support. The space C_{00} is automatically contained in the space $\mathscr{B}$ due to (B-1)(i). The space BC is contained in $\mathscr{B}$ under the additional axiom (C).

(C). If a uniformly bounded sequence $\{\phi^n(\theta)\}$ in C_{00} converges to a function $\phi(\theta)$ uniformly on every compact set of $(-\infty, 0]$, then $\phi \in \mathscr{B}$ and $\lim_{n\to\infty} |\phi^n - \phi|_{\mathscr{B}} = 0$.

In fact, BC is continuously imbedded into $\mathscr{B}$; put

$$\|\phi\|_\infty = \sup\{|\phi(\theta)| : \theta \leqslant 0\} \qquad \text{for} \quad \phi \in BC.$$

The following result is found in [6].

Lemma 2.1. *If the phase space* $\mathscr{B}$ *satisfies the axiom* (C), *then* $BC \subset \mathscr{B}$ *and there is a constant* $J > 0$ *such that* $|\phi|_{\mathscr{B}} \leqslant J\,\|\phi\|_\infty$ *for all* $\phi \in BC$.

For each $b \in E$, define a constant function $\bar{b}$ by $\bar{b}(\theta) = b$ for $\theta \in (-\infty, 0]$; then $|\bar{b}|_{\mathscr{B}} \leqslant J\,|b|$ from Lemma 2.1. Define operators $S(t)\colon \mathscr{B} \to \mathscr{B}$, $t \geqslant 0$, as follows:

$$[S(t)\,\phi](\theta) = \begin{cases} \phi(0) & -t \leqslant \theta \leqslant 0, \\ \phi(t+\theta) & \theta \leqslant -t. \end{cases}$$

Let $S_0(t)$ be the restriction of $S(t)$ to $\mathscr{B}_0 := \{\phi \in \mathscr{B} : \phi(0) = 0\}$. If $x\colon R \to E$ is continuous on $[\sigma, \infty)$ and $x_\sigma \in \mathscr{B}$, we take a function $y\colon R \to E$ defined by $y(t) = x(t)$, $t \geqslant \sigma$; $y(t) = x(\sigma)$, $t \leqslant \sigma$. From Lemma 2.1, $y_t \in \mathscr{B}$ for $t \geqslant \sigma$, and x_t is decomposed as

$$x_t = y_t + S_0(t-\sigma)[x_\sigma - \overline{x(\sigma)}] \qquad \text{for} \quad t \in [0, \infty). \tag{2.1}$$

Using Lemma 2.1 and (2.1), we have an inequality

$$|x_t|_{\mathscr{B}} \leqslant J \sup\{|x(s)| : \sigma \leqslant s \leqslant t\} + |S_0(t-\sigma)[x_\sigma - \overline{x(\sigma)}]|_{\mathscr{B}}. \tag{2.2}$$

The phase space $\mathscr{B}$ is called a fading memory space [6] if the axiom (C) holds and $S_0(t)\,\phi \to 0$ as $t \to \infty$ for each $\phi \in \mathscr{B}_0$. If $\mathscr{B}$ is such a space, then $\|S_0(t)\|$ is bounded for $t \geqslant 0$ by the Banach Steinhaus theorem, and

$$|x_t|_{\mathscr{B}} \leqslant J \sup\{|x(s)| : \sigma \leqslant s \leqslant t\} + M\,|x_\sigma|_{\mathscr{B}}, \tag{2.3}$$

where $M = (1 + HJ) \sup_{t \geqslant 0} \|S_0(t)\|$. As a result, if $x(t)$ is E-bounded, then x_t is $\mathscr{B}$-bounded. In addition, if $\|S_0(t)\| \to 0$ as $t \to \infty$, then $\mathscr{B}$ is called a uniform fading memory space. It is shown in [6, p. 190], that the phase space $\mathscr{B}$ is a uniform fading memory space if and only if the axiom (C) holds and $K(t)$ is bounded and $\lim_{t\to\infty} M(t)=0$ in the axiom (B-1).

We state a property for the α-measure of noncompactness of bounded sets in $\mathscr{B}$. For a bounded subset Ω of a Banach space X we define

$$a_X(\Omega)=\inf\{d>0\colon \Omega \text{ has a finite cover of diameter } <d\},$$

which is called the Kuratowskii measure (for brevity, α-measure) of noncompactness of Ω. The subscript X is usually omitted. Obviously, $\alpha(\Omega)=0$ if and only if Ω is relatively compact in X; $\alpha(k\Omega)=|k|\,\alpha(\Omega)$ for a scalar k; and $\alpha(\Omega_1+\Omega_2)\leqslant\alpha(\Omega_1)+\alpha(\Omega_2)$.

If $L\colon X\to X$ takes bounded sets into bounded sets, we define

$$\alpha(L)=\inf\{k>0\colon \alpha(L\Omega)\leqslant k\alpha(\Omega) \text{ for all bounded sets } \Omega\subset X\}.$$

Then L is a compact operator if and only if $\alpha(L)=0$. If L is a bounded linear operator, then $\alpha(L)\leqslant\|L\|$, and the essential spectral radius of L is given as

$$r_e(L)=\lim_{n\to\infty}\alpha(L^n)^{1/n},$$

which we call the Nussbaum formula, see [11]. The following fact follows easily from the definition of essential spectrum.

LEMMA 2.2. *Let L be a bounded linear operator on a Banach space X. If $|\lambda|>r_e(L)$, then the range $R(\lambda I-L)$ is closed in X and* dim $N(\lambda I-L)$ *is finite.*

Since $\alpha(L^n)\leqslant\alpha(L)^n$, we have $r_e(L)\leqslant\alpha(L)$. Lemma 2.2 is a revised version of Ambrosetti's theorem [1]: if $|\lambda|>\alpha(L)$, then the range $R(\lambda I-L)$ is closed in X. Lemma 2.2 implies the following result.

COROLLARY 2.3. *If $r_e(L)<1$, then $I-L\in\Phi_+(X)$.*

Let $C[a,b]$ be the set of all continuous functions from $[a,b]$ into E with the supremum norm. Let $\mathscr{X}$ be a set of functions $x\colon(-\infty,\sigma+a)\to E$, $0<a\leqslant\infty$, such that $x_\sigma\in\mathscr{B}$ and x is continuous on $[\sigma,\sigma+a)$. We will use the notations

$$\mathscr{X}(t)=\{x(t)\in E\colon x\in\mathscr{X}\},\qquad \mathscr{X}_t=\{x_t\in\mathscr{B}\colon x\in\mathscr{X}\}\quad\text{for}\quad t\in[\sigma,\sigma+a),$$

$$\mathscr{X}\,|\,[c,d]=\{x\,|\,[c,d]\in C[c,d]\colon x\in\mathscr{X}\},$$

where $\sigma\leqslant c\leqslant d<\sigma+a$ and $x\,|\,[c,d]$ stands for the restriction of x to $[c,d]$.

LEMMA 2.4. *Let $t \geqslant \sigma$. If $\mathscr{X}_\sigma$ and $\mathscr{X} \mid [\sigma, t]$ are bounded in $\mathscr{B}$ and $C[\sigma, t]$, respectively, then the following relations hold.*

(1) $H^{-1}\alpha(\mathscr{X}(t)) \leqslant \alpha(\mathscr{X}_t) \leqslant K(t-\sigma)\,\alpha(\mathscr{X} \mid [\sigma, t]) + M(t-\sigma)\,\alpha(\mathscr{X}_\sigma)$.

(2) *If the phase space $\mathscr{B}$ satisfies the axiom* (C), *then*

(i) $\alpha(\mathscr{X}_t) \leqslant J\alpha(\mathscr{X} \mid [\sigma, t]) + (1+HJ)\,\alpha(S_0(t-\sigma))\,\alpha(\mathscr{X}_\sigma)$,

and, in addition, if $\mathscr{X}_\sigma$ is bounded in BC, then

(ii) $\alpha(\mathscr{X}_t) \leqslant J \max\{\alpha_{BC}(\mathscr{X}_\sigma), \alpha(\mathscr{X} \mid [\sigma, t])\}$,

where J is as in Lemma 2.1.

Proof. The assertion (1) is proved in [16]. By the decomposition (2.1), we have

$$\begin{aligned}
\alpha(\mathscr{X}_t) &\leqslant \alpha(\{y_t \in \mathscr{B} : x \in \mathscr{X}\}) + \alpha(\{S_0(t-\sigma)[x_\sigma - \overline{x(\sigma)}] : x \in \mathscr{X}\}) \\
&\leqslant J\alpha(\mathscr{X} \mid [\sigma, t]) + \alpha(S_0(t-\sigma))\,\alpha(\{x_\sigma - \overline{x(\sigma)} : x \in \mathscr{X}\}) \\
&\leqslant J\alpha(\mathscr{X} \mid [\sigma, t]) + \alpha(S_0(t-\sigma))[\alpha(\mathscr{X}_\sigma) + J\alpha(\mathscr{X}(\sigma))] \\
&\leqslant J\alpha(\mathscr{X} \mid [\sigma, t]) + (1+HJ)\,\alpha(S_0(t-\sigma))\,\alpha(\mathscr{X}_\sigma).
\end{aligned}$$

Next, we consider the semilinear functional differential equation of the following form

$$\text{(S)} \qquad \frac{dx}{dt} = Ax(t) + F(t, x_t), \qquad t > \sigma.$$

We will estimate the solution $x(t, \sigma, \phi)$ for Eq. (S) with $x_\sigma = \phi \in \mathscr{B}$. In addition to (H-1), we assume that F has the following property:

(H_0) $F: R \times \mathscr{B} \to E$ is continuous and satisfies a locally Lipschitz condition with respect to the second variable, and there are continuous functions $n, f: R \to [0, \infty)$ such that

$$|F(t, \psi)| \leqslant n(t)\,|\psi|_{\mathscr{B}} + f(t) \qquad \text{for all} \quad (t, \psi) \in R \times \mathscr{B}.$$

If $x: (-\infty, \sigma + a) \to E$ is continuous on $[\sigma, \sigma + a)$, and if

$$\begin{cases} x(t) = T(t-\sigma)\,\phi(0) + \displaystyle\int_\sigma^t T(t-s)\,F(s, x_s)\,ds, & \sigma \leqslant t < \sigma + a, \\ x_\sigma = \phi \in \mathscr{B}, \end{cases}$$

then it is called a (mild) solution of Eq. (S) with $x_\sigma = \phi$. We denote by $x(t, \sigma, \phi)$ such a solution.

PROPOSITION 2.5. *The solution of Eq.* (S) *with* $x_\sigma = \phi \in \mathscr{B}$ *exists uniquely on* $[\sigma, \infty)$.

The proof follows from [5, 17, 19]. To derive the estimate of solutions for Eq. (S), we recall Gronwall's inequality of the following type. Suppose that $u(t)$, $a(t)$, $b(t)$ and $G(t)$ are nonnegative functions, $u(t)$, $a(t)$, $b(t)$ are continuous, $G(t)$ is locally absolutely continuous, and that

$$u(t) \leqslant a(t)\, G(t) + a(t) \int_\sigma^t b(s)\, u(s)\, ds \qquad \text{for} \quad t \geqslant \sigma.$$

Then,

$$\begin{aligned} u(t) \leqslant a(t)\, G(\sigma) \exp\left[\int_\sigma^t a(r)\, b(r)\, dr\right] \\ + a(t) \int_\sigma^t G'(s) \exp\left[\int_s^t a(r)\, b(r)\, dr\right] ds \qquad \text{for} \quad t \geqslant \sigma. \end{aligned}$$

PROPOSITION 2.6. *Let* $\|T(t)\| \leqslant M_w e^{wt}$, $t \geqslant 0$, *for the* C_0*-semigroup* $T(t)$ *in* (H). *Then the solution* $x(t, \sigma, \phi)$ *of Eq.* (S) *is estimated as*

$$\begin{aligned} &|x_t(\sigma, \phi)|_{\mathscr{B}} \\ &\leqslant K(t-\sigma) \int_\sigma^t N(t, s, \sigma)\, f(s)\, ds \\ &\quad + |\phi|_{\mathscr{B}} \left\{ M(t-\sigma) + K(t-\sigma) \left[HN(t, \sigma, \sigma) + \int_\sigma^t N(t, s, \sigma)\, n(s)\, M(s-\sigma)\, ds \right] \right\}, \end{aligned}$$

where

$$N(t, s, \sigma) = M_w \exp \int_s^t [w_+ + M_w n(r)\, K(r-\sigma)]\, dr, \qquad \sigma \leqslant s \leqslant t,$$

and $w_+ = \max\{w, 0\}$. *In particular, if* $f(t) \equiv 0$, *then there is a locally integrable function* $m(\cdot, \sigma)\colon [\sigma, \infty) \to [0, \infty)$ *such that*

$$|x_t(\sigma, \phi)|_{\mathscr{B}} \leqslant m(t, \sigma)\, |\phi|_{\mathscr{B}} \qquad \textit{for all} \quad t \in [\sigma, \infty).$$

If $\mathscr{B}$ *is a fading memory space, and if* $|F(t, \psi)| \leqslant \ell\, |\psi|_{\mathscr{B}}$, $\ell > 0$, *in* (H_0), *then the solution* $x(t, \sigma, \phi)$ *of Eq.* (S) *is estimated as*

$$|x_t(\sigma, \phi)|_{\mathscr{B}} \leqslant |\phi|_{\mathscr{B}}\, (HJM_w + M) \exp\{(M_w \ell J + w_+)(t-\sigma)\},$$

where $M = (1 + HJ) \sup_{t \geqslant 0} \|S_0(t)\|$.

Outline of the Proof. The proof is standard, but the computation is rather complicated; so we keep a record of the outline. Set $u(t) = \sup\{|x(s)| : \sigma \leqslant s \leqslant t\}$ for the solution $x(t) = x(t, \sigma, \phi)$. From the axiom (B-1) and the condition (H_0), it follows that, for $t \geqslant \sigma$,

$$|x(t)| \leqslant M_w e^{w(t-\sigma)} |\phi(0)| + \int_\sigma^t M_w e^{w(t-s)} \times \{n(s)[K(s-\sigma)\, u(s) + M(s-\sigma)|\phi|_{\mathscr{B}}] + |f(s)|\}\, ds.$$

If we replace w by w_+, then the right side becomes a nondecreasing function of $t \geqslant \sigma$. Thus $u(t)$ satisfies Gronwall's inequality mentioned before, where $a(t) = M_w e^{(w_+)t}$, $b(t) = e^{-(w_+)t} n(t)\, K(t-\sigma)$ and

$$G(t) = e^{-(w_+)\sigma} |\phi(0)| + \int_\sigma^t e^{-(w_+)s} \{n(s)\, M(s-\sigma)\, |\phi|_{\mathscr{B}} + |f(s)|\}\, ds.$$

We have the desired estimate by combining again the inequality in (B-1)(ii) with the estimate of $u(t)$ derived from Gronwall's inequality.

If $\mathscr{B}$ is a fading memory space, we can take $K(t-\sigma) = J$, $M(t-\sigma) = M$ as in (2.3). If $n(t) = \ell$, $f(t) = 0$, then $N(t, s, \sigma) = M_w e^{v(t-s)}$, where $v = (w_+) + M_w \ell J$, and

$$|x_t|_{\mathscr{B}} \leqslant |\phi|_{\mathscr{B}} \left\{ M + J \left[H M_w e^{v(t-\sigma)} + \frac{M_w \ell M}{(w_+) + M_w \ell J} (e^{v(t-\sigma)} - 1) \right] \right\}.$$

Replacing the denominator of the fraction in the right side by $M_w \ell J$, we obtain the estimate in the proposition.

For the solution $x(t, \sigma, \phi)$ of Eq. (S) defined on $[\sigma, \infty)$, the solution operator $U(t, \sigma)$, $t \geqslant \sigma$, is defined by $U(t, \sigma)\, \phi = x_t(\sigma, \phi)$ for $\phi \in \mathscr{B}$, and it is decomposed as follows. Define

$$[K(t, \sigma)\, \phi](\theta) = \begin{cases} \displaystyle\int_\sigma^{t+\theta} T(t+\theta-s)\, F(s, x_s(\sigma, \phi))\, ds & t+\theta \geqslant \sigma, \\ 0, & t+\theta \leqslant \sigma. \end{cases}$$

Then $U(t, \sigma) = \hat{T}(t-\sigma) + K(t, \sigma)$, $t \geqslant \sigma$, and the family $\hat{T}(t)$, $t \geqslant 0$, is a C_0-semigroup on $\mathscr{B}$. If $\|T(t)\| \leqslant M_w e^{wt}$, $t \geqslant 0$, and if $\mathscr{B}$ is a fading memory space, then

$$\|\hat{T}(t)\| \leqslant (J M_w H + M)\, e^{w_+ t}. \tag{2.4}$$

In fact, from the definition of $\hat{T}(t)$ and (2.3), it follows that

$$|\hat{T}(t)\,\phi|_{\mathscr{B}} \leqslant J \sup\{|T(s)\,\phi(0)|: 0 \leqslant s \leqslant t\} + M|\phi|_{\mathscr{B}}$$
$$\leqslant JM_w He^{w_+ t}\,|\phi|_{\mathscr{B}} + M\,|\phi|_{\mathscr{B}}.$$

3. PROPERTIES OF THE OPERATOR $K(t, \sigma)$

We consider the condition that $K(t, \sigma)$ is compact as well as the estimate of its norm. For a continuous function $u: [a, b] \to E$ we put

$$(T * u)(t) = \int_a^t T(t-s)\,u(s)\,ds \qquad \text{for} \quad t \in [a, b],$$

and for a subset $\mathscr{U} \subset C[a, b]$ we put $T * \mathscr{U} = \{T * u: u \in \mathscr{U}\}$. The following lemma is found in [18].

Lemma 3.1. *Let $\mathscr{U}$ be a bounded set in $C[a, b]$, and $T(t)$ a C_0-semigroup on E. Then*

$$\alpha((T * \mathscr{U}) \mid [a, t]) \leqslant \gamma_T \sup_{a \leqslant \tau \leqslant t} \alpha((T * \mathscr{U})(\tau))$$

for all $t \in [a, b]$, where $\gamma_T = \limsup_{t \to 0} \|T(t)\|$. In particular, if $T(t)$ is a C_0-contraction semigroup on E, then

$$\alpha((T * \mathscr{U}) \mid [a, t]) = \sup_{a \leqslant \tau \leqslant t} \alpha((T * \mathscr{U})(\tau)).$$

The following result is found in [8].

Lemma 3.2. *Let W be a countable set of strongly measurable functions $x^n: [a, b] \to E$, $n = 1, 2, \ldots$. Assume that there exists a function $\mu(t)$, which is integrable on $[a, b]$, such that $|x^n(t)| \leqslant \mu(t)$ for all $x^n \in W$ and for a.a.$t \in [a, b]$. Then $\alpha(W(t))$ is integrable on $[a, b]$ and*

$$\alpha\left(\left\{\int_a^b x^n(t)\,dt: x^n \in W\right\}\right) \leqslant 2 \int_a^b \alpha(W(t))\,dt.$$

A C_0-semigroup $T(t)$ is called a C_0-compact semigroup for $t > t_0$ if for every $t > t_0$, $T(t)$ is a compact operator. $T(t)$ is called compact if it is compact for $t > 0$. Using Lemma 3.1 and Lemma 3.2, we have the following result.

PROPOSITION 3.3. *If $T(t)$ is a C_0-compact semigroup on E or if $F(t, \cdot)$ is a compact operator for each $t \in R$, then $\alpha(K(t, \sigma)) = 0$ for all $t \geqslant \sigma$; that is, $K(t, \sigma)$ is a compact operator.*

Proof. If $t = \sigma$, this is trivial. Let $t > \sigma$, and Ω be any bounded set in $\mathscr{B}$. From Proposition 2.6 it follows that $x_s(\sigma, \phi)$ is bounded on $[\sigma, t]$ uniformly for $\phi \in \Omega$. Take the family $\mathscr{F}$ consisting of functions $f(s) = F(s, x_s(\sigma, \phi))$ with parameter $\phi \in \Omega$. From Lemma 2.4 we have $\alpha(K(t, \sigma)\,\Omega) \leqslant K(t-\sigma)$ $\alpha((T * \mathscr{F}) \mid [\sigma, t])$; from lemma 3.1 we have

$$\alpha((T * \mathscr{F}) \mid [\sigma, t]) \leqslant \gamma_T \sup_{\sigma \leqslant \tau \leqslant t} \alpha((T * \mathscr{F})(\tau)).$$

Hence it is sufficient to see that, for any $t > \sigma$, $\alpha((T * \mathscr{F})(t)) = 0$. The equivalent condition is that, for any sequence $\{\phi^n\}$ in Ω, the sequence $\{(T * f^n)(t)\}$ contains a convergent subsequence, or $\alpha(\{(T * f^n)(t)\}) = 0$, where $f^n(s) = F(s, x_s(\sigma, \phi^n))$. Indeed, this follows from Lemma 3.2 together with the fact that $\alpha(\{T(t-s)\, F(s, x_s(\sigma, \phi)) : \phi \in \Omega\}) = 0$ for $s \in [0, t)$, because of the assumption in the proposition.

We consider the case that $F(t, \phi) = B(t, \phi)$ in Eq. (P_ωL); then, $K(t, 0)$ is a linear operator. Set

$$\|B\|_\infty := \sup\{\|B(t)\| : 0 \leqslant t < \infty\} = \sup\{\|B(t)\| : 0 \leqslant t \leqslant \omega\},$$

where $\|B(t)\|$ is the operator norm of $B(t, \cdot)$. It is finite since $B(t, \phi)$ is continuous.

PROPOSITION 3.4. *Suppose that $\|T(t)\| \leqslant M_w e^{wt}$ for $t \geqslant 0$ and that $\mathscr{B}$ is a fading memory space. Then, for Eq. ($P_\omega L_0$), we have that*

$$\|K(t, 0)\| \leqslant a \int_0^t e^{(a + w_+)(t-s)} \|\hat{T}(s)\|\, ds$$

for $t \geqslant 0$, where $a = JM_w \|B\|_\infty$, $w_+ = \max\{w, 0\}$.

Proof. Observe that we can write, for $t + \theta \geqslant 0$,

$$[K(t, 0)\, \phi](\theta) = \int_0^{t+\theta} T(t+\theta-s)\, B(s, \hat{T}(s)\, \phi + K(s, 0)\, \phi)\, ds.$$

Since $\mathscr{B}$ is a fading memory space, we have that

$$|K(t, 0)\, \phi|_{\mathscr{B}} \leqslant J \sup_{0 \leqslant \tau \leqslant t} \int_0^\tau M_w e^{w(\tau - s)} \|B\|_\infty (\|\hat{T}(s)\|\, |\phi|_{\mathscr{B}} + |K(s, 0)\, \phi|_{\mathscr{B}})\, ds.$$

If we replace w by w_+, then the function of τ defined by the integral becomes a nondecreasing function. Thus we have that

$$|K(t, 0)\,\phi|_{\mathscr{B}} \leqslant JM_w \|B\|_\infty \int_0^t e^{w_+(t-s)}(\|\hat{T}(s)\|\;|\phi|_{\mathscr{B}} + |K(s, 0)\,\phi|_{\mathscr{B}})\,ds.$$

Put

$$f(t) = \int_0^t e^{-sw_+}\|\hat{T}(s)\|\,ds\,|\phi|_{\mathscr{B}}, \qquad u(t) = e^{-tw_+}\,|K(t, 0)\,\phi|_{\mathscr{B}}.$$

Then

$$u(t) \leqslant af(t) + a\int_0^t u(s)\,ds$$

for $t \geqslant 0$. From Gronwall's inequality, it follows that

$$u(t) \leqslant af(t) + a^2\int_0^t e^{a(t-s)}f(s)\,ds.$$

Since $\|\hat{T}(t)\|$ is lower semi-continuous, it is measurable. Hence we have that

$$\begin{aligned}\int_0^t e^{a(t-s)}f(s)\,ds &= \int_0^t\int_r^t e^{-as}\,ds e^{-rw_+}\,\|\hat{T}(r)\|\,dr\,|\phi|_{\mathscr{B}} \\ &= a^{-1}\int_0^t e^{a(t-r)}e^{-rw_+}\,\|\hat{T}(r)\|\,dr\,|\phi|_{\mathscr{B}} - a^{-1}f(t).\end{aligned}$$

Thus,

$$u(t) \leqslant a\int_0^t e^{a(t-r)}e^{-rw_+}\,\|\hat{T}(r)\|\,dr\,|\phi|_{\mathscr{B}},$$

or

$$|K(t, 0)\,\phi|_{\mathscr{B}} \leqslant a\int_0^t e^{(a+w_+)(t-s)}\,\|\hat{T}(s)\|\,ds\,|\phi|_{\mathscr{B}},$$

which implies our inequality.

Combining this estimate with (2.4) we have the following one.

COROLLARY 3.5. *Under the assumptions in Proposition* 3.4,

$$\|K(t,0)\| \leqslant (JM_w H + M)\, e^{w+t}[\exp(JM_w \|B\|_\infty t) - 1] \qquad \text{for} \quad t \geqslant 0.$$

LEMMA 3.6. *Let a and w be positive constants, and let $f, u \colon [0, c] \to R$ be nonnegative continuous functions. Suppose that $f(t)$ is a nondecreasing function in t and that $u(t)$ satisfies the inequality*

$$u(t) \leqslant a \sup_{o \leqslant \tau \leqslant t} \int_0^\tau e^{-w(\tau - s)} u(s)\, ds + f(t).$$

If $w > a$, then

$$u(t) \leqslant wf(t)/(w - a).$$

Proof. Set $v(t) := \sup\{u(s) \colon 0 \leqslant s \leqslant t\}$. From the condition for $u(t)$ we obtain that

$$v(t) \leqslant av(t) \int_0^t e^{-ws}\, ds + f(t) \leqslant av(t)/w + f(t).$$

This implies the desired inequality for $u(t)$.

PROPOSITION 3.7. *Suppose that $\|T(t)\| \leqslant M_w e^{-wt}$, $M_w, w > 0$, for $t \geqslant 0$ and that $\mathscr{B}$ is a fading memory space. Then, for Eq. $(P_\omega L_0)$*

$$\|K(t,0)\| \leqslant JM_w \|B\|_\infty (JM_w H + M) \times \min\{w^{-1} e^{JM_w \|B\|_\infty t}, 1/(w - JM_w \|B\|_\infty)\},$$

for $t \geqslant 0$ provided that $w > JM_w \|B\|_\infty$.

Proof. Set $a = JM_w \|B\|_\infty$ and $b = JM_w H + M$. Then we have

$$|K(t,0)\,\phi|_{\mathscr{B}} \leqslant a \sup_{0 \leqslant \tau \leqslant t} \int_0^\tau e^{-w(\tau - s)} |K(s,0)\,\phi|_{\mathscr{B}}\, ds + a \sup_{0 \leqslant \tau \leqslant t} \int_0^\tau e^{-w(\tau - s)} |\hat{T}(s)\,\phi|_{\mathscr{B}}\, ds.$$

From (2.4) we have

$$a \sup_{0 \leqslant \tau \leqslant t} \int_0^\tau e^{-w(\tau - s)} |\hat{T}(s)\,\phi|_{\mathscr{B}}\, ds \leqslant a \sup_{0 \leqslant \tau \leqslant t} \int_0^\tau e^{-w(\tau - s)} b\, |\phi|_{\mathscr{B}}\, ds \leqslant ab\, |\phi|_{\mathscr{B}}/w.$$

If we set $u(t)=|K(t,0)\,\phi|_{\mathscr{B}}$, then

$$u(t)\leqslant a\sup_{0\leqslant\tau\leqslant t}\int_0^\tau e^{-w(\tau-s)}u(s)\,ds+ab\,|\phi|_{\mathscr{B}}/w.$$

From this, we have

$$u(t)\leqslant a\int_0^t u(s)\,ds+ab\,|\phi|_{\mathscr{B}}/w.$$

Using Gronwall's inequality, we get $u(t)\leqslant abe^{at}|\phi|_{\mathscr{B}}/w$. On the other hand, by Lemma 3.6, we get $u(t)\leqslant ab\,|\phi|_{\mathscr{B}}/(w-a)$ if $w>a$. Summarizing the above results, we have the estimate in the proposition.

Remark. Combining Proposition 3.7 and Corollary 3.5, we have, for $w>JM_w\|B\|_\infty$,

$$\|K(t,0)\|\leqslant b\min\left\{e^{JM_w\,\|B\|_\infty\,t}-1,\ w^{-1}JM_w\,\|B\|_\infty\,e^{JM_w\,\|B\|_\infty\,t},\ \frac{JM_w\,\|B\|_\infty}{w-JM_w\,\|B\|_\infty}\right\},$$

where $b=JM_wH+M$.

4. α-MEASURE OF THE OPERATOR $\hat{T}(T)$

In this section we compute the α-measure of the operator $\hat{T}(t)$ as well as $r_e(\hat{T}(t))$. For a bounded set $\mathscr{H}$ of $C([a,b],E)$ we put

$$\omega(\delta;t,\mathscr{H})=\sup\{|g(\tau)-g(s)|:\tau,s\in[t-\delta,t+\delta],\ g\in\mathscr{H}\}$$

$$\omega(t,\mathscr{H})=\inf\{\omega(\delta;t,\mathscr{H}):\delta>0\}=\lim_{\delta\to0+}\omega(\delta;t,\mathscr{H})$$

$$\omega(\mathscr{H})=\sup_{a\leqslant t\leqslant b}\omega(t,\mathscr{H}).$$

We note that $\mathscr{H}$ is equicontinuous on $[a,b]$ if and only if $\omega(\mathscr{H})=0$. The following result can be found in [12].

LEMMA 4.1. *Let $\mathscr{H}$ be a bounded subset of $C([a,b],E)$. Then*

$$\max\{(1/2)\,\omega(\mathscr{H}),\ \sup_{a\leqslant t\leqslant b}\alpha(\mathscr{H}(t))\}\leqslant\alpha(\mathscr{H})\leqslant2\omega(\mathscr{H})+\sup_{a\leqslant t\leqslant b}\alpha(\mathscr{H}(t)).$$

For a subset $D \subset E$ we denote by $T(\cdot)\,D$ the family of functions $T(t)\,x$ defined for $t \in [0, \infty)$ with a parameter $x \in D$, and by $T(\cdot)\,D \mid [a, b]$ its restriction to $[a, b]$.

LEMMA 4.2. *Let $D \subset E$ be bounded.*

(1) *If $T(t)$ is a C_0-semigroup on E,*

$$\alpha(T(\cdot)\,D \mid [a, b]) \leqslant \sup_{a \leqslant s \leqslant b} \|T(s)\|\, \alpha(D), \qquad b > a \geqslant 0.$$

(2) *Let $T(t)$ be a C_0-semigroup on E such that $T(t)\,x \in \mathscr{D}(A)$ for all $x \in E$ and $t > 0$. If $b > \varepsilon > 0$, then*

(i) $\alpha(T(\cdot)\,D \mid [\varepsilon, b]) \leqslant \sup_{\varepsilon \leqslant s \leqslant b} \alpha(T(s))\, \alpha(D)$.

(ii) $\alpha(T(\cdot)\,D \mid [0, b])$
$$\leqslant \max\{\sup_{0 \leqslant s \leqslant \varepsilon} \|T(s)\|, \sup_{\varepsilon \leqslant s \leqslant b} \alpha(T(s))\}\, \alpha(D).$$

(3) *Let $T(t)$ be a C_0-compact semigroup on E. If $0 < \varepsilon < b$, then*

(i) $\alpha(T(\cdot)\,D \mid [\varepsilon, b]) = 0$.

(ii) $\alpha(T(\cdot)\,D \mid [0, b]) \leqslant \delta_T \alpha(D)$,

where $\delta_T := \max\{1, \gamma_T\} = \max\{1, \limsup_{t \to 0} \|T(t)\|\}$.

Proof. The assertion (1) is derived directly from the definition of $\alpha(T(\cdot)\,D \mid [\varepsilon, b])$.

Set $\mathscr{H} = \alpha(T(\cdot)\,D \mid [\varepsilon, b])$. We show that $\omega(t, \mathscr{H}) = 0$ for $t > 0$ if $T(t)$ has the property in (2) or (3). It is clear in the case (3). Consider the case (2). Put $M := \sup\{|x| : x \in D\}$. Then M is finite, and $|T(s)\,x - T(t)\,x| \leqslant \|T(s) - T(t)\|\, M$ for $s, t \geqslant 0$, $x \in D$. Since $T(t)\,x \in D(A)$, $t > 0$, $x \in E$, it follows that $T(t)$ is continuous for $t > 0$ in the uniform operator topology, cf. [13, p. 52]. This implies that $\omega(t, \mathscr{H}) = 0$ for $t > 0$.

Now from Lemma 4.1 it follows that $\alpha(\mathscr{H}) = \sup_{\varepsilon \leqslant s \leqslant b} \alpha(\mathscr{H}(s))$. Since $\alpha(\mathscr{H}(s)) \leqslant \alpha(T(s))\, \alpha(D)$, we have the properties (2)(i) and (3)(i). Since

$$\alpha(T(\cdot)\,D \mid [0, b]) \leqslant \max\{\alpha(T(\cdot)\,D \mid [0, \varepsilon]), \alpha(T(\cdot)\,D \mid [\varepsilon, b])\},$$

we obtain the properties (2)(ii) and (3)(ii).

We have the following result about $\alpha(\hat{T}(t))$ from Lemmas 2.4 and 4.2 as well as the fact that $\alpha(\Omega(0)) \leqslant H\alpha(\Omega)$ for a bounded set $\Omega \subset \mathscr{B}$, where $\Omega(0) = \{\phi(0) : \phi \in \Omega\}$.

LEMMA 4.3.

(1) *Let* $T(t)$ *be a* C_0*-semigroup on* E.

(i) $\alpha(\hat{T}(t)) \leqslant HK(t) \sup_{0 \leqslant s \leqslant t} \|T(s)\| + M(t)$ *for* $t > 0$.

(ii) *If the phase space* $\mathscr{B}$ *satisfies the axiom* (C), *then*

$$\alpha(\hat{T}(t)) \leqslant HJ \sup_{0 \leqslant s \leqslant t} \|T(s)\| + (1 + HJ)\, \alpha(S_0(t)) \qquad \text{for} \quad t > 0.$$

(2) *Let* $T(t)$ *be a* C_0*-semigroup on* E *such that* $T(t)\, x \in \mathscr{D}(A)$ *for all* $x \in E$ *and* $t > 0$.

(i) $\alpha(\hat{T}(t)) \leqslant HK(t) \max\{\sup_{0 \leqslant s \leqslant \varepsilon} \|T(s)\|,\ \sup_{\varepsilon \leqslant s \leqslant t} \alpha(T(s))\} + M(t)$ *for* $t > \varepsilon > 0$.

(ii) *If the phase space* $\mathscr{B}$ *satisfies the axiom* (C), *then, for* $t > \varepsilon > 0$,

$$\alpha(\hat{T}(t)) \leqslant HJ \max\{\sup_{0 \leqslant s \leqslant \varepsilon} \|T(s)\|, \sup_{\varepsilon \leqslant s \leqslant t} \alpha(T(s))\} + (1 + HJ)\, \alpha(S_0(t)).$$

(3) *Let* $T(t)$ *be a* C_0*-compact semigroup on* E.

(i) $\alpha(\hat{T}(t)) \leqslant HK(t)\, \delta_T + M(t)$ *for* $t > 0$.

(ii) *If the phase space* $\mathscr{B}$ *satisfies the axiom* (C), *then*

$$\alpha(\hat{T}(t)) \leqslant HJ\delta_T + (1 + HJ)\, \alpha(S_0(t)) \qquad \text{for} \quad t > 0.$$

For a bounded set $\Omega \subset \mathscr{B}$ and for $t > 0$, we have that $\hat{T}(t)\, \Omega = \hat{T}(t/2)(\hat{T}(t/2)\, \Omega)$. Thus it follows that

$$\alpha(\hat{T}(t)\, \Omega) \leqslant K(t/2)\, \alpha(T(\cdot)\, \Omega(0) \mid [t/2, t]) + M(t/2)\, \alpha(\hat{T}(t/2)\, \Omega).$$

From the several estimates of $\alpha(T(\cdot)\, \Omega(0) \mid [t/2, t])$ and $\alpha(\hat{T}(t/2)\, \Omega)$ given by Lemma 4.2 and Lemma 4.3(1) and (2), we have the following two propositions.

PROPOSITION 4.4. *Let* $T(t)$ *be a* C_0*-semigroup on* E.

(1) $\alpha(\hat{T}(t)) \leqslant HK(t/2) \sup_{t/2 \leqslant s \leqslant t} \|T(s)\| + HM(t/2)\, K(t/2) \sup_{0 \leqslant s \leqslant t/2} \|T(s)\| + M^2(t/2)$.

(2) *If the phase space* $\mathscr{B}$ *satisfies the axiom* (C), *then*

$$\alpha(\hat{T}(t)) \leqslant HJ \sup_{t/2 \leqslant s \leqslant t} \|T(s)\| + (1 + HJ)\, HJ \sup_{0 \leqslant s \leqslant t/2} \|T(s)\|\, \alpha(S_0(t/2)) + (1 + HJ)^2\, \alpha^2(S_0(t/2)).$$

PROPOSITION 4.5. *Let $T(t)$ be a C_0-semigroup on E such that $T(t)\, x \in \mathscr{D}(A)$ for all $x \in E$ and $t > 0$.*

(1) $\alpha(\hat{T}(t)) \leqslant HK(t/2) \sup_{t/2 \leqslant s \leqslant t} \alpha(T(s))$
$\qquad + HM(t/2)\, K(t/2) \sup_{0 \leqslant s \leqslant t/2} \|T(s)\| + M^2(t/2)$.

(2) *If the phase space $\mathscr{B}$ satisfies the axiom* (C), *then*

$$\alpha(\hat{T}(t)) \leqslant HJ \sup_{t/2 \leqslant s \leqslant t} \alpha(T(s)) + (1 + HJ)\, HJ \sup_{0 \leqslant s \leqslant t/2} \|T(s)\|\, \alpha(S_0(t/2))$$

$$+ (1 + HJ)^2\, \alpha^2(S_0(t/2)).$$

If $T(t)$ is a C_0-compact semigroup for $t > t_0$, where $t_0 \geqslant 0$, we have that

$$\begin{aligned}\alpha(\hat{T}(t)\, \Omega) &\leqslant K(t - t_0 - \varepsilon)\, \alpha(T(\cdot)\, \Omega(0) \mid [t_0 + \varepsilon, t]) \\ &\quad + M(t - t_0 - \varepsilon)\, \alpha(\hat{T}(t_0 + \varepsilon)\, \Omega) \\ &= M(t - t_0 - \varepsilon)\, \alpha(\hat{T}(t_0 + \varepsilon)\, \Omega),\end{aligned}$$

as long as $t_0 < t_0 + \varepsilon < t$. From this, we have the following Propositions 4.6 and 4.7. To represent them, we prepare a definition. For a given locally bounded function $g\colon [0, \infty) \to [0, \infty)$, we define a function $\tilde{g}\colon [0, \infty) \to [0, \infty)$ as

$$\tilde{g}(t) = \begin{cases} \limsup\limits_{s \to t-0} g(s), & t > 0, \\ \max\{g(0), \limsup\limits_{s \to 0+} g(s)\}, & t = 0. \end{cases}$$

If $g(t) = \alpha(S_0(t))$, then we denote $\tilde{g}(t)$ by $\tilde{\alpha}(S_0(t))$.

PROPOSITION 4.6. *If $T(t)$ is a C_0-compact semigroup on E, then the following relations hold for $t > 0$:*

(1) $\alpha(\hat{T}(t)) \leqslant C_1 \widetilde{M}(t)$.

(2) *If the phase space $\mathscr{B}$ satisfies the axiom* (C), *then* $\alpha(\hat{T}(t)) \leqslant C_2 \tilde{\alpha}(S_0(t))$,

where $C_1 = HK(0)\, \delta_T + \widetilde{M}(0)$, $C_2 = (1 + JH)\, C_1$.

PROPOSITION 4.7. *Suppose that $T(t)$ is a C_0-compact semigroup for $t > t_0$ on E.*

(1) $\alpha(\hat{T}(t)) \leqslant \hat{C}_1(t_0)\, \tilde{M}(t-t_0),\ t > t_0$.

(2) *if $\mathscr{B}$ satisfies the axiom* (C), *then*

$$\alpha(\hat{T}(t)) \leqslant \hat{C}_2(t_0)\, \tilde{\alpha}(S_0(t-t_0)), \qquad t > t_0,$$

where

$$\hat{C}_1(t_0) = \limsup_{\varepsilon \to 0+} \{HK(t_0+\varepsilon) \sup_{0 \leqslant s \leqslant t_0+\varepsilon} \|T(s)\| + M(t_0+\varepsilon)\}$$

$$\hat{C}_2(t_0) = \limsup_{\varepsilon \to 0+} \{HJK(t_0+\varepsilon) \sup_{0 \leqslant s \leqslant t_0+\varepsilon} \|T(s)\|$$

$$+ (1+HJ)\, \alpha(S_0(t_0+\varepsilon))\}(1+HJ).$$

Suppose that a function $g\colon [0, \infty) \to [0, \infty)$ is locally bounded, and submultiplicative (that is, $g(t+s) \leqslant g(t)\, g(s)$ for $t, s \geqslant 0$). Then it is well known that $\lim_{t\to\infty} t^{-1} \log g(t) = \inf_{t>0} t^{-1} \log g(t)$, which may be $-\infty$, but not be $+\infty$. This quantity is called the type number of the function $g(t)$. Notice that the function $\tilde{g}$ defined as before is also submultiplicative and locally bounded, and that the type number of the function g is negative if and only if that of the function $\tilde{g}$ is so. We denote respectively by

$$\tau,\ \tau^{\nu},\ \hat{\tau},\ \hat{\tau}^{\nu},\ \beta_0,\ \tilde{\beta}_0,\ \beta_0^{\nu},\ \beta,\ \beta^{\nu} \text{ and } \tilde{\mu}$$

the type numbers of the functions

$$\alpha(T(t)),\ \|T(t)\|,\ \alpha(\hat{T}(t)),\ \|\hat{T}(t)\|,\ \alpha(S_0(t)),\ \tilde{\alpha}(S_0(t)),\ \|S_0(t)\|,$$
$$\alpha(S(t)),\ \|S(t)\| \text{and } \tilde{M}(t),$$

provided that $T(t)$ is a C_0-semigroup on E and that $M(t)$ is submultiplicative. Notice that

$$r_e(\hat{T}(t)) = \exp(\hat{\tau} t), \qquad t > 0,$$

by using the Nussbaum formula. Thus, if $\hat{\tau} < 0$, then $I - \hat{T}(\omega) \in \Phi_+(\mathscr{B})$, see Corollary 2.3. Now we shall show several conditions that $\hat{\tau}$ is negative.

THEOREM 4.8. (1) *Let $T(t)$ be a C_0-semigroup on E. If τ^{ν} is negative and if the phase space $\mathscr{B}$ is a uniform fading memory space, then $\hat{\tau} < 0$.*

(2) *Let $T(t)$ be a C_0-semigroup on E such that $T(t)\, x \in \mathscr{D}(A)$ for all $x \in E$ and $t > 0$. If $\tau^{\nu} + \beta_0^{\nu} < 0$ and $\tau < 0$, and if the phase space $\mathscr{B}$ is a uniform fading memory space, then $\hat{\tau} < 0$.*

(3) *Let $T(t)$ be a C_0-compact semigroup on E or a C_0-compact semigroup for $t > t_0$ on E.*

(i) *If* $\lim_{t \to \infty} M(t) = 0$, *then* $\hat{\tau} < 0$.

(ii) *If* $M(t)$ *is submultiplicative, then* $\hat{\tau} \leqslant \tilde{\mu}$.

(iii) *If the phase space* $\mathscr{B}$ *satisfies the axiom* (C), *then* $\hat{\tau} \leqslant \tilde{\beta}_0$.

Proof. (1) Since τ^ν and β_0 are negative, there are some $M_w \geqslant 1$ and w, where $0 > w > \max\{\tau^\nu, \beta_0\}$, such that $\max\{\|T(t)\|, \alpha(S_0(t))\} \leqslant M_w e^{wt}$ for $t \geqslant 0$. Hence, from the assertion (2) in Proposition 4.4 we have

$$\alpha(\hat{T}(t)) \leqslant \{HJM_w + (1+HJ)\, HJM_w^2 + (1+HJ)^2\, M_w^2\}\, e^{wt/2},$$

which implies that $\hat{\tau} \leqslant w/2 < 0$.

(2) In view of Proposition 4.5, we can easily prove assertion (2) by using the same manner as (1).

(3) From Proposition 4.6 we have that $\alpha(\hat{T}(t-\sigma)^n) = \alpha(\hat{T}(n(t-\sigma))) \leqslant C_1 \tilde{M}(n(t-\sigma))$. If $\lim_{t \to \infty} M(t) = 0$, then $\lim_{t \to \infty} \tilde{M}(t) = 0$; consequently, $C_1 \tilde{M}(n(t-\sigma)) < 1$ for sufficiently large n. It follows that $\inf_{n \geqslant 1} \log \alpha(\hat{T}(n(t-\sigma))) < 0$, which implies $\hat{\tau} < 0$: the assertion (i) is proved. The rest of the proof is obvious from Proposition 4.7.

Before ending of this section, we consider relationships among the quantities β_0^ν, β_0, β^ν and β. To do so, we assume the following axiom in addition to the axioms (B-1) and (B-2).

(B-3). $|\phi^1 - \phi^2|_{\mathscr{B}} = 0$ for ϕ^1, ϕ^2 in $\mathscr{B}$ if and only if $\phi^1(\theta) = \phi^2(\theta)$ for $\theta \in (-\infty, 0]$.

We denote by $P_\sigma(L)$ and $E_\sigma(L)$ the point spectrum and the essential spectrum, respectively, for a bounded linear operator L on a Banach space X. Let $\mathscr{C}$ be the subspace of constant functions, on $(-\infty, 0]$, lying in the space $\mathscr{B}$. If $\mathscr{B}$ satisfies the axiom (C), then $\mathscr{C} = \{\bar{c} : c \in E\}$. However, it is possible that $\mathscr{C} = \{0\}$ in general. From the axiom (B-1)(ii), $\bar{c} \in \mathscr{C}$ is a nonzero vector if and only if $c \neq 0$. We notice that, by the definition of the essential spectrum, a point in the spectrum outside the essential spectrum lies in the point spectrum.

PROPOSITION 4.9. *Let* $t > 0$, *and suppose that* $\mathscr{B}$ *satisfies the additional axiom* (B-3). *Then,* $S_0(t)$ *has not a nonzero eigenvalue; hence,* $\sigma(S_0(t)) \backslash \{0\} = E_\sigma(S_0(t)) \backslash \{0\}$. *A possible, nonzero eigenvalue of* $S(t)$ *is only one; and* $P_\sigma(S(t)) \backslash \{0\} = \{1\}$ *if and only if* $\mathscr{C}$ *is a nontrivial subspace of* $\mathscr{B}$. *In this case* $N((I - S(t))^k) = \mathscr{C}$, $k = 1, 2, \ldots$, *and hence, if* $\dim \mathscr{C} = \infty$, *then* $\sigma(S(t)) \backslash \{0\} = E_\sigma(S(t)) \backslash \{0\}$.

Proof. Suppose that $\mu \in P_\sigma(S_0(t))$, $\mu \neq 0$. Then there exists a $\phi \in \mathscr{B}_0$, $\phi \neq 0$, such that $S_0(t)\,\phi = \mu\phi$. Since $S_0(nt)\,\phi = S_0^n(t)\,\phi = \mu^n \phi$, $n = 1, 2, \ldots$, it follows that $\phi = \mu^{-n} S_0(nt)\,\phi$. Applying the axiom (B-3), we see that $\phi = 0$: this is a contradiction. Hence we have the first assertion of the proposition.

Suppose that $\mu \in P_\sigma(S(t))$, $\mu \neq 0$. Then there exists a $\phi \in \mathscr{B}$, $\phi \neq 0$, such that $S(t)\,\phi = \mu\phi$. If $\phi(0) = 0$, then $\phi \in \mathscr{B}_0$, $S_0(t)\,\phi = \mu\phi$; hence $\mu \in P_\sigma(S_0(t))$, which is impossible, Thus $\phi(0) \neq 0$. Since $(S(t)\,\phi)(0) = \mu\phi(0)$ from the axiom (B-1)(ii), we have $\phi(0) = \mu\phi(0)$, which implies $\mu = 1$. Then $\phi = S(nt)\,\phi$, $n = 1, 2, \ldots$; hence $\phi = \overline{\phi(0)}$ by (B-3). Since $\phi(0) \neq 0$, we have $\overline{\phi(0)} \neq 0$; thus $\mathscr{C}$ is a nontrivial subspace of $\mathscr{B}$. Conversely, if $\phi = \bar{c}$ lies in $\mathscr{B}$ for some $c \in E$, $c \neq 0$, then it is clear that $1 \in P_\sigma(S(t))$.

Finally we show that $N((I - S(t))^k) = N(I - S(t))$, $k = 1, 2, \ldots$, in the case $\mathscr{C}$ is a nontrivial subspace of $\mathscr{B}$. It suffices to show this for $k = 2$. Suppose that $(I - S(t))^2\,\phi = 0$. Then $(I - S(t))\,\phi = \bar{c}$ for some $c \in E$. Taking the value at $\theta = 0$, we have that $\phi(0) - \phi(0) = c$: that is, $c = 0$, or $\bar{c} = 0$. Thus $(I - S(t))\,\phi = 0$.

COROLLARY 4.10. *Suppose that $\mathscr{B}$ satisfies the additional axiom* (B-3). *Then $\beta_0^\nu = \beta_0$; hence, $\|S_0(t)\| \to 0$ as $t \to \infty$ if and only if $\alpha(S_0(t)) \to 0$ as $t \to \infty$. If* $\dim \mathscr{C} = \infty$, *then $\beta^\nu = \beta \geqslant 0$.*

Combining this corollary and [6, Lemma 4.3.1], we have the following result.

COROLLARY 4.11. *Suppose that $\mathscr{B}$ satisfies the additional axiom* (B-3) *and that* $\dim E < \infty$. *Then $\beta_0^\nu = \beta_0 = \beta \leqslant \beta^\nu$.*

5. MAIN THEOREM IN THE GENERAL PHASE SPACE $\mathscr{B}$

In the beginning, we present the formula for the radius of the essential spectrum of the solution operator of Eq. $(\mathrm{P}_\omega \mathrm{L}_0)$.

PROPOSITION 5.1. *Let $T(t)$ be a C_0-compact semigroup on E, or $B(t, \cdot)$ be a compact operator for each $t \in R$. If $U(t, \sigma)$ is the solution operator of Eq.* (L_0), *then*

$$r_e(U(t, \sigma)) = r_e(\hat{T}(t - \sigma)) = \exp(\hat{\tau}(t - \sigma)), \qquad t > \sigma.$$

Proof. From the assumption and Proposition 3.3, it follows that $K(t, \sigma)$ is compact. Hence we have $\alpha(U(t, \sigma)^n) = \alpha(\hat{T}(t - \sigma)^n)$, $n = 1, 2, \ldots$, which implies the formula in the theorem from the Nussbaum formula.

Now we present main theorems deduced from the compactness of $K(\omega, 0)$ as well as the estimates of $\alpha(\hat{T}(\omega))$ obtained in Section 4.

THEOREM 5.2. *Let $T(t)$ be a C_0-compact semigroup on E, or $B(t, \cdot)$ be a compact operator for each $t \in R$. If $\hat{\tau} < 0$ and Eq.* $(\mathrm{P}_\omega \mathrm{L})$ *has a $\mathscr{B}$-bounded solution, then it has an ω-periodic solution.*

Proof. Set $P = U(\omega, 0)$, the solution operator of Eq. $(\mathrm{P}_\omega \mathrm{L}_0)$. From Proposition 5.1 the condition $\hat{\tau} < 0$ implies that $r_e(P) < 1 : I - P \in \Phi_+(\mathscr{B})$.

Define a linear affine mapping $V: \mathscr{B} \to \mathscr{B}$ as $V\psi = P\psi + x_\omega(0, 0, F)$ for $\psi \in \mathscr{B}$. From the definition of the operator P we have

$$V\psi = x_\omega(0, \psi, 0) + x_\omega(0, 0, F) = x_\omega(0, \psi, F).$$

In general, $V^k\psi = x_{k\omega}(0, \psi, F)$ for $k = 1, 2, \ldots$. Let $x(t) = x(t, 0, \phi, F)$ be a $\mathscr{B}$-bounded solution of Eq. $(\mathrm{P}_\omega \mathrm{L})$; that is, $\sup\{|x_t|_{\mathscr{B}} : t \geqslant 0\} < \infty$. Since $V^k\phi = x_{k\omega}(0, \phi, F)$, the sequence $\{V^k\phi\}$ is a bounded sequence of $\mathscr{B}$. Applying Theorem 1.1, we see that V has a fixed point in $\mathscr{B}$. It is not difficult to show that the solution of Eq. $(\mathrm{P}_\omega \mathrm{L})$ starting from this fixed point is an ω-periodic solution.

If the phase space $\mathscr{B}$ is a fading memory space, then E-bounded solutions become $\mathscr{B}$-bounded solutions. Furthermore, if the phase space $\mathscr{B}$ is a uniform fading memory space, then $\lim_{t \to \infty} M(t) = 0$, see Section 2.

THEOREM 5.3. *Assume that the phase space $\mathscr{B}$ is a uniform fading memory space and that at least one of the following conditions is satisfied:*

(1) *$T(t)$ is a C_0-compact semigroup on E.*

(2) *$B(t, \cdot)$ is a compact operator for each $t \in R$ and $\tau^\nu < 0$.*

(3) *$T(t)$ is a C_0-compact semigroup for $t > t_0$ on E and $B(t, \cdot)$ is a compact operator for each $t \in R$.*

(4) *$T(t)$ is a C_0-semigroup on E such that $T(t)\, x \in \mathscr{D}(A)$ for all $x \in E$ and $t > 0$, $B(t, \cdot)$ is a compact operator for each $t \in R$, and $\tau^\nu + \beta_0^\nu < 0$, $\tau < 0$.*

If Eq. $(\mathrm{P}_\omega \mathrm{L})$ *has an E-bounded solution, then it has an ω-periodic solution.*

Proof. If Condition (1) or (3) holds, then $\hat{\tau} < 0$ from Theorem 4.8 (3)(i) since $\lim_{t \to \infty} M(t) = 0$. If Condition (2) holds, then $\hat{\tau} < 0$ from Theorem 4.8 (1). If Condition (4) holds, then $\hat{\tau} < 0$ from Theorem 4.8 (2). Thus the theorem follows from Theorem 5.2.

Remark. We state a remark on Henriquez's paper [5] for Eq. (S). If $T(t)$ is a C_0-compact semigroup on E and if $\mathscr{B}$ is a uniform fading memory space, then it follows from Proposition 4.6 (2) that $C_2\tilde{\alpha}(S_0(\omega)) < 1$ for sufficiently large ω. Thus $P := U(\omega, 0)$ is condensing, where $U(\omega, 0)$ is the solution operator of Eq. (S) for which $F(t, \psi)$ is ω-periodic in t. Suppose that there exist a closed, bounded and convex subset $Z \subset \mathscr{B}$ such that $P(Z) \subset Z$. From Sadovskii's fixed point theorem, it then follows that there exists a point $\phi \in Z$ such that $P(\phi) = \phi$. Namely, there exists a periodic solution of Eq. (S) under these conditions.

Finally, we consider Eq. (P_ωL) with $A = 0$:

$$\frac{dx(t)}{dt} = B(t, x_t) + F(t).$$

For this equation we can take $S(t)$ as the C_0-semigroup $T(t)$ in (H). Suppose that $\mathscr{B}$ satisfies the axiom (B-3) and that $\dim \mathscr{C} = \infty$. Then Proposition 4.9 and Corollary 4.10 say that the type number β of $\alpha(S(t))$ is not negative, and that $\dim N((I - S(t))^k) = \dim \mathscr{C} = \infty$; that is, $I - S(\omega) \notin \Phi_+(\mathscr{B})$. Thus it is still an open problem whether this equation has property (BP) or not even if every $B(t, \cdot)$, $t \in R$, or $K(\omega, 0)$ is a compact operator.

6. CLOSED RANGE CONDITIONS

We will show in the following that the semi-Fredholm property of $I - \hat{T}(\omega)$ is inherited from the one of $I - T(\omega)$ at least if $\mathscr{B} = UC_g$ and it is a uniform fading memory space. Throughout this section, we assume that the phase space $\mathscr{B}$ always satisfies the axioms (B-1), (B-2), (B-3), and (C). We begin with the following observation.

LEMMA 6.1. *If $T(t)$ is a C_0-semigroup on E, then a function ϕ of $N(I - \hat{T}(\omega))$ is an ω-periodic continuous function given by $\phi(\theta) = T(\theta + n\omega)\, \phi(0)$, $\theta \in [-n\omega, 0]$, $n = 1, 2, ...,$ where $\phi(0) \in N(I - T(\omega))$, and*

$$\dim N(I - \hat{T}(\omega)) = \dim N(I - T(\omega)).$$

Proof. Suppose that $\hat{T}(\omega)\phi = \phi$. Since $[\hat{T}(\omega)\, \phi](\theta) = \phi(\omega + \theta)$ for $\omega + \theta \leqslant 0$, it follows that $\phi(\omega + \theta) = \phi(\theta)$ for $\theta \leqslant -\omega$; that is, $\phi(\theta)$ is ω-periodic on $(-\infty, 0]$. Since $\hat{T}(n\omega) = \hat{T}(\omega)^n$, $n = 0, 1, 2, ...,$ we have that $\hat{T}(n\omega)\, \phi = \phi$. On the other hand, if $-n\omega \leqslant \theta \leqslant 0$, then $[\hat{T}(n\omega)\, \phi](\theta) = T(n\omega + \theta)\, \phi(0)$; hence, $T(n\omega + \theta)\, \phi(0) = \phi(\theta)$ for $-n\omega \leqslant \theta \leqslant 0$ and $\phi(\theta)$ is continuous on $[-n\omega, 0]$. Set $a = \phi(0)$ and $x(t) = T(t)\, a$, $t \geqslant 0$. Then $x(t) = \phi(t - n\omega)$ as long as $0 \leqslant t \leqslant n\omega$. Since n may be arbitrary, we can regarde that $x(t)$ is ω-periodic and continuous in $(-\infty, \infty)$, and $\phi = x_0$. Since $x(\omega) = x(0)$, it follows that $T(\omega)\, a = a$; that is, $a \in N(I - T(\omega))$.

Conversely, if $a \in N(I - T(\omega))$, then $T(t + \omega)\, a = T(t)\, T(\omega)\, a = T(t)\, a$, $t \geqslant 0$; that is, $T(t)\, a$ is ω-periodic in $[0, \infty)$. Let $x(t)$ be the ω-periodic extension of $T(t)\,a$ to $(-\infty, \infty)$, and set $\phi = x_0$. From the axiom (C) we see that ϕ belongs to $\mathscr{B}$. Then it is obvious that $\hat{T}(\omega)\,\phi = \phi$. Moreover, the space $N(I - T(\omega))$ is mapped bijectively onto the space $N(I - \hat{T}(\omega))$. Therefore, the proof is complete.

We will solve the equation $(I - \hat{T}(\omega))\,\phi = \psi$ to prove that the range $R(I - \hat{T}(\omega))$ is closed.

PROPOSITION 6.2. *The functions* $\phi, \psi \in \mathscr{B}$ *satisfy the equation* $(I - \hat{T}(\omega))\,\phi = \psi$ *if and only if*

(1) $(I - T(\omega))\,\phi(0) = \psi(0)$.

(2) $\phi(\theta) = \sum_{j=0}^{k-1} \psi(\theta + j\omega) + T(\theta + k\omega)\,\phi(0)$, $\theta \in I_k$, $k = 1, 2, \ldots$, *where* $I_k = [-k\omega, -(k-1)\,\omega]$.

Proof. Suppose that $(I - \hat{T}(\omega))\,\phi = \psi$. Then

$$\psi(\theta) = \begin{cases} \phi(\theta) - T(\theta + \omega)\,\phi(0) & \theta \in I_1 \\ \phi(\theta) - \phi(\theta + \omega) & \theta \in I_k, \quad k \geqslant 2. \end{cases}$$

Taking the value at $\theta = 0$ in the first equation, we obtain the condition (1). Solving this equation with respect to $\phi(\theta)$ on I_k successively for $k = 1, 2, \ldots$, we have the representation of $\phi(\theta)$ in the condition (2) in the above. The value $\phi(-k\omega)$ is well defined for $k \geqslant 1$ because of the condition $(I - T(\omega))\,\phi(0) = \psi(0)$.

Conversely, if $\phi, \psi \in \mathscr{B}$ have the properties (1) and (2), then it follows immediately that $(I - \hat{T}(\omega))\,\phi = \psi$.

Let the null space $N(I - T(\omega))$ be of finite dimension. Then there exists a closed subspace M of E such that $E = N \oplus M$, where $N = N(I - T(\omega))$, cf. [15]. Let S_M be the restriction of $I - T(\omega)$ to M. Then $S_M : M \to R(I - T(\omega))$ is a continuous, bijective, linear operator. Thus there is the inverse operator S_M^{-1} of S_M; if $R(I - T(\omega))$ is closed, then S_M^{-1} is continuous.

Let ψ be a function in $\mathscr{B}$ such that $\psi(0) \in R(I - T(\omega))$. Since $R(S_M) = R(I - T(\omega))$, $S_M^{-1}\psi(0)$ is well defined and $(I - T(\omega))\,S_M^{-1}\psi(0) = \psi(0)$. Define a function $U\psi : (-\infty, 0] \to E$ pointwise by

$$[U\psi](\theta) = \sum_{j=0}^{k-1} \psi(\theta + j\omega) + T(\theta + k\omega)\,S_M^{-1}\psi(0), \qquad \theta \in I_k, \qquad (6.1)$$

for $k = 1, 2, \ldots$. Notice that $D(U) = \{\psi \in \mathscr{B} : \psi(0) \in R(I - T(\omega))\}$.

PROPOSITION 6.3. *Suppose that* $\dim N(I-T(\omega))<\infty$. *Then*

$$R(I-\hat{T}(\omega))=\{\psi\colon U\psi\in\mathscr{B}\}.$$

Proof. Suppose that $U\psi\in\mathscr{B}$, and set $\phi:=U\psi$. Taking the value at $\theta=0$ in (6.1) with $k=1$, we have that $\phi(0)=\psi(0)+T(\omega)\,S_M^{-1}\psi(0)$. However, since

$$\psi(0)+T(\omega)\,S_M^{-1}\psi(0)=(I-T(\omega))\,S_M^{-1}\psi(0)+T(\omega)\,S_M^{-1}\psi(0)=S_M^{-1}\psi(0),$$

the function ϕ is represented as in (2) in the preceding proposition, and the equation (1) also holds. Hence $(I-\hat{T}(\omega))\,\phi=\psi$. Conversely, suppose that $\psi\in R(I-\hat{T}(\omega))$, that is, $(I-\hat{T}(\omega)))\,\phi=\psi$ for some $\phi\in\mathscr{B}$. Then the equations (1), (2) in the preceding proposition hold. Set $\phi(0)=\phi_M(0)+\phi_N(0)$, where $\phi_M(0)\in M$, $\phi_N(0)\in N(I-T(\omega))$. Then we have $\phi(\theta)=U\psi(\theta)+T(\theta+k\omega)\,\phi_N(0)$ for $\theta\in I_k$, $k=1,2,...$. Since $\phi_N(0)\in N(I-T(\omega))$, the function $\chi(\theta):=T(\theta+k\omega)\,\phi_N(0)$, $\theta\in I_k$, $k\geqslant 1$, defines an ω-periodic continuous function; hence, $\chi\in\mathscr{B}$ because of the axiom (C). Therefore $U\psi=\phi-\chi$ is a function in $\mathscr{B}$, as desired.

To prove that $R(I-\hat{T}(\omega))$ is closed, we prepare well known theorems for the closed range. Let X and Y be Banach spaces. The first one is as follows. Let T be a one to one, closed liner operator on X into Y. Then a necessary and sufficient condition that $R(T)$ is closed is that there exists a constant c such that $|x|\leqslant c\,|Tx|$ for $x\in D(T)$, see [15, Theorem 5.1, p. 70]. The second one is derived from this by taking the quotient space $D(T)/N(T)$. If N is a closed subspace of X, the quotient space X/N is the collection of the subsets $[x]=x+N:=\{x+y\colon y\in N\}$ for all $x\in X$. It is a Banach space with the norm $|[x]|:=\inf\{|x+y|\colon y\in N\}$. Let T be a closed linear operator on X into Y which may not be one to one. Then $R(T)$ is closed if and only if there is a constant c such that $|[x]|\leqslant c\,|Tx|$ for all $x\in D(T)$.

We consider the case where T is a bounded linear operator on X into Y such that $N:=N(T)$ is of finite dimension. Then there exists a closed subspace Z of X such that $X=N\oplus Z$. Now we define a map $\pi\colon Z\to X/N$ as $\pi(z)=[z]$ for $z\in Z$. Since π is a continuous isomorphism, π^{-1} is also continuous from the closed graph theorem. Therefore, there is a constant k_Z such that

$$|[z]|\leqslant|z|\leqslant k_Z\,|[z]|\qquad\text{for}\quad z\in Z.$$

Let P_Z be the projection onto N along Z.

LEMMA 6.4. *Let T be a bounded linear operator on X, and N, Z, P_Z, k_Z be as in the above; then the following statements are equivalent.*

(1) *$R(T)$ is closed.*

(2) *There exists a constant c_Z such that*

$$|(I-P_Z)\,x| \leqslant c_Z\,|Tx| \qquad \textit{for} \quad x \in X.$$

(3) *There exists a constant c such that*

$$|[x]| \leqslant c\,|Tx| \qquad \textit{for} \quad x \in X.$$

In fact, (2) *implies* (3) *as $c = c_Z$, while* (3) *implies* (2) *as $c_Z = ck_Z$.*

Proof. It is sufficient to show that the conditions (2) and (3) are equivalent. see [15, Lemma 6.1, p. 125]. If the condition (2) holds, then, for $x \in X$,

$$|[x]| = |[(I-P_Z)\,x]| \leqslant |(I-P_Z)\,x| \leqslant c_Z\,|Tx|.$$

Conversely, if the condition (3) holds, then

$$|(I-P_Z)\,x| \leqslant k_Z\,|[(I-P_Z)\,x]| \leqslant ck_Z\,|T(I-P_Z)\,x| = ck_Z\,|Tx|.$$

THEOREM 6.5. *Suppose that $I - T(\omega) \in \Phi_+(E)$, and take the quotient space $\mathscr{B}/N(I-\hat{T}(\omega))$. If there exists a positive constant c such that*

$$|U\psi|_{\mathscr{B}} \leqslant c\,|\psi|_{\mathscr{B}} \qquad \textit{as long as} \quad U\psi \in \mathscr{B}, \tag{6.2}$$

where U is defined by (6.1), *then*

$$|[\phi]| \leqslant c\,|(I-\hat{T}(\omega))\,\phi|_{\mathscr{B}} \qquad \textit{for} \quad \phi \in \mathscr{B}:$$

as a result, $R(I-\hat{T}(\omega))$ is closed.

Furthermore, this condition for U is a necessary and sufficient condition that $R(I-\hat{T}(\omega))$ is closed provided that the space $\mathscr{B}$ has the following property: if $|\phi^n - \phi|_{\mathscr{B}} \to 0$ as $n \to \infty$, then $\phi^n(\theta)$ converges to $\phi(\theta)$ at each $\theta \in (-\infty, 0]$.

Proof. Suppose that $(I-\hat{T}(\omega))\,\phi = \psi$. Then $\psi \in R(I-\hat{T}(\omega))$; hence $U\psi \in \mathscr{B}$, because of Proposition 6.3, and $\psi = (I-\hat{T}(\omega))\,U\psi$. Since $[\phi] = [U\psi]$, using the condition in the theorem we have

$$|[\phi]| \leqslant |U\psi|_{\mathscr{B}} \leqslant c\,|\psi|_{\mathscr{B}} = c\,|(I-\hat{T}(\omega))\,\phi|_{\mathscr{B}}.$$

Therefore, we have the first part of the theorem.

Set $D = \{U\psi : \psi \in R(I-\hat{T}(\omega))\}$. Then D is a subspace of $\mathscr{B}$ by Proposition 6.3. Let F be the restriction of $I-\hat{T}(\omega)$ to D. Then the operator $F: D \to R(I-\hat{T}(\omega))$ have the following properties: $FU\psi = \psi$ for

$\psi \in R(I-\hat{T}(\omega))$, $N(F)=\{0\}$, $R(F)=R(I-\hat{T}(\omega))$, and F is a bounded linear operator. The condition for U in the theorem is rewritten as $|\phi| \leqslant c\,|F\phi|$ for $\phi \in D$. If D is known to be closed, then from Lemma 6.4 (3), this condition for F is a necessary and sufficient condition that $R(F)=R(I-\hat{T}(\omega))$ is closed. But it is not clear that D is closed. In this reason, we go to another direction. In fact, there is a similar result for the closed linear operator mentioned before Lemma 6.4. That is, if F is a closed operator, then $R(F)=R(I-\hat{T}(\omega))$ is closed if and only if there is a positive constant c such that $|\phi|_{\mathscr{B}} \leqslant c\,|F\phi|_{\mathscr{B}}$ for all $\phi \in D$. Thus it suffices to show that F is a closed operator. Even if F is a bounded linear operator, this is not trivial since the domain D is not known to be a closed subspace. By using the additional property for $\mathscr{B}$, we will show directly that F is a closed operator. To do so, suppose that a sequence $\phi^n := U\psi^n$, $n=1, 2, ...$ in D converges to a function ϕ in $\mathscr{B}$ and the sequence $F\phi^n=\psi^n$ converges to a function ψ in $\mathscr{B}$. From the assumption in the theorem it follows that $\phi^n(\theta) \to \phi(\theta)$, $\psi^n(\theta) \to \psi(\theta)$ as $n \to \infty$ at each $\theta \in (-\infty, 0]$. Since $\psi^n(0) \to \psi(0)$ as $n \to \infty$, and since $R(I-T(\omega))$ is closed, we have that $\psi(0) \in R(I-T(\omega))$. Thus $\psi \in D(U)$, and $U\psi^n(\theta) \to U\psi(\theta)$ as $n \to \infty$ at each $\theta \in (-\infty, 0]$. Notice that we use here that fact S_M^{-1} is continuous. Therefore $U\psi(\theta)=\phi(\theta)$ for all $\theta \in (-\infty, 0]$. Since $\phi \in \mathscr{B}$, it follows that $\psi \in R(I-\hat{T}(\omega))$, $\phi = U\psi \in D$ and $F\phi=\psi$.

Take the phase space as $\mathscr{B}=UC_g$, the set of continuous functions, $\phi(\theta)$, such that $\phi(\theta)/g(\theta)$ is bounded and uniformly continuous on $(-\infty, 0]$ with the norm

$$\|\phi\| = \sup\{|\phi(\theta)|/g(\theta) : \theta \leqslant 0\},$$

where $g(\theta)$ is a positive continuous function such that $g(\theta) \to \infty$ as $\theta \to -\infty$. Then $\|S_0(t)\| = \sup_{s \leqslant 0} g(s)/g(s-t)$, and it is a uniform fading memory space if and only if it is a fading memory space, cf. [6, p. 191]. We will check the condition (6.2) in the above theorem for $\mathscr{B}=UC_g$.

THEOREM 6.6. *If $\mathscr{B}=UC_g$ is a uniform fading memory space and if $I-T(\omega) \in \Phi_+(E)$, then $I-\hat{T}(\omega) \in \Phi_+(\mathscr{B})$.*

Proof. Since $\mathscr{B}=UC_g$ is a uniform fading memory space, there are $M_0 \geqslant 1$ and $\varepsilon_0 > 0$ such that $\|S_0(t)\| \leqslant M_0 e^{-\varepsilon_0 t}$ for $t \geqslant 0$. Namely,

$$\|S_0(t)\| = \sup_{s \leqslant 0} g(\theta)/g(\theta - t) = \sup_{s \leqslant -t} g(s+t)/g(s) \leqslant M_0 e^{-\varepsilon_0 t}.$$

Suppose that $\psi(0) \in R(I-T(\omega))$, $U\psi \in \mathscr{B}$, $\psi \in \mathscr{B}$. Let $\theta \in [-k\omega, -(k-1)\,\omega]$, $k \geqslant 1$. Then we have

$$\begin{aligned}\frac{1}{g(\theta)}\left|\sum_{j=0}^{k-1}\psi(\theta+j\omega)\right| &\leqslant \sum_{j=0}^{k-1}\frac{g(\theta+j\omega)}{g(\theta)}\frac{|\psi(\theta+j\omega)|}{g(\theta+j\omega)}\\ &\leqslant \sum_{j=0}^{k-1}\|S_0(j\omega)\|\;\|\psi\|\\ &\leqslant \|\psi\|\sum_{j=0}^{k-1}M_0e^{-\varepsilon_0 j\omega}\leqslant \|\psi\|\;M_0/(1-e^{-\varepsilon_0\omega}).\end{aligned}$$

On the other hand, since S_M^{-1} is continuous, we have that

$$|T(\theta+k\omega)\,S_M^{-1}\psi(0)|/g(\theta)\leqslant \sup\{\|T(t)\|:0\leqslant t\leqslant\omega\}\,\|S_M^{-1}\|\;\|\psi\|.$$

Summarizing these inequalities, the constant c in (6.2) is estimated as

$$\|U\psi\|\leqslant (M_0/(1-e^{-\varepsilon_0\omega})+\sup\{\|T(t)\|:0\leqslant t\leqslant\omega\}\,\|S_M^{-1}\|)\;\|\psi\|.$$

Therefore, $I-\hat{T}(\omega)\in\Phi_+(\mathscr{B})$.

EXAMPLE. Let γ be a negative constant, and take the phase space UC_γ, the set of continuous functions, ψ, such that $e^{-\gamma\theta}\psi(\theta)$ is bounded and uniformly continuous for $\theta\in(-\infty,0]$, and set $\|\psi\|=\sup\{e^{-\gamma\theta}|\psi(\theta)|:\theta\leqslant 0\}$. Then $\|S_0(t)\|=e^{\gamma t}\to 0$ as $t\to\infty$. Hence if $\mathscr{B}=UC_\gamma$ and $\gamma<0$, then Theorem 6.6 holds.

We give an example for the space $\mathscr{B}$ on which $R(I-\hat{T}(\omega))$ is not closed for any C_0-semigroup $T(t)$ on E such that $N(I-T(\omega))=\{0\}$. Let BUC be the set of all bounded, uniformly continuous functions from $(-\infty,0]$ into E with the supremum norm. This space has the properties (B-1), (B-2), (B-3), (C) and the one required for the phase space in the end of Theorem 6.5. But we have the following result.

PROPOSITION 6.7. *Suppose that $N(I-T(\omega))=\{0\}$, and take the space BUC as the phase space of $\hat{T}(t)$. Then $R(I-\hat{T}(\omega))$ is not closed.*

Proof. From Theorem 6.5, it suffices to show that there exists a sequence $\{\phi^n\}$ in BUC such that $|\phi^n|_{\mathscr{B}}\equiv 1$, and $\lim_{n\to\infty}|(I-\hat{T}(\omega))\,\phi^n|_{\mathscr{B}}=0$. Let e be a unit vector of E; that is, $|e|_{\mathscr{B}}=1$, and define $x^n(t)$, $n=1,2,...,$ as

$$x^n(t)=\begin{cases} e & t\leqslant -n\omega\\ (-t/n\omega)\,e & -n\omega\leqslant t\leqslant 0\\ 0 & t\geqslant 0.\end{cases}$$

Set $\phi^n=x_0^n$, $n=1,2,....$ Since $\phi^n(0)=0$, we have $[\hat{T}(\omega)(\phi^n)](\theta)=0$ for $\theta\in[-\omega,0]$; in other words, $\hat{T}(\omega)\,\phi^n=S_0(\omega)\,\phi^n$. Thus it follows that

$(I - \hat{T}(\omega))\,\phi^n = \phi^n - S_0(\omega)\,\phi^n$; hence, $|(I - \hat{T}(\omega))\,\phi^n|_{\mathscr{B}} = 1/n \to 0$ as $n \to 0$. Clearly, $|\phi^n|_{\mathscr{B}} \equiv 1$. Thus this is a desired sequence.

7. MAIN THEOREMS IN THE PHASE SPACE UC_g

We obtain main theorems in the uniform fading memory space from the semi-Fredholm condition in the preceding section. The first one is concerned with the compact property.

THEOREM 7.1. *Assume that $\mathscr{B} = UC_g$ is a uniform fading memory space and at least one of the following conditions is satisfied*:

(1) *$T(t)$ is a C_0-compact semigroup on E.*

(2) *$B(t, \cdot)$ is a compact operator for each $t \in R$ and $I - T(\omega) \in \Phi_+(E)$.*

If Eq. $(\mathrm{P}_\omega \mathrm{L})$ *has a E-bounded solution, then it has an ω-periodic solution.*

Proof. The proof easily follows from Theorem 6.5, the assertion (1) in Theorem 1.2 and Proposition 3.3.

Remark. If $\mathscr{B} = UC_g$, then Theorem 5.3 (2) and (3) are direct consequences of Theorem 7.1 (2). Indeed, it is trivial for Theorem 5.3 (2); if C_0-semigroup $T(t)$ is compact for $t > t_0$, we have $\tau = -\infty$ and hence $r_e(T(t)) = 0$. This implies $I - T(\omega) \in \Phi_+(E)$. Therefore Theorem 5.3 (3) is derived from Theorem 7.1 (2).

Remark. If $\mathscr{B} = UC_g$, Theorem 7.1 and Theorem 5.3 are generalizations of the result by Hino, Murakami, and Yoshizawa stated in the Introduction.

We need some preparation to apply our result for the perturbation theory with respect to the norm condition of $K(\omega, 0)$. Let $T: X \to Y$ be a bounded linear operator such that $\dim N(T) < \infty$ and that the condition (2) in Lemma 6.4 holds for some constant c_Z. Then the following result is well known for the perturbation, cf. [15, Theorems 6.3 and 6.4, p. 128].

LEMMA 7.2. *Let T be the operator in the above. If $S: X \to Y$ is a bounded linear operator satisfying $\|S\| < 1/2c_Z$, then $T + S \in \Phi_+(X, Y)$ and*

$$\dim N(T + S) \leqslant \dim N(T).$$

To apply this for our case, we modify it as follows.

PROPOSITION 7.3. *Suppose that $T: X \to X$ is a bounded linear operator such that* $\dim N(T) = n < \infty$, *and that, there exists a positive constant c for which $|[x]| \leqslant c\,|Tx|$ for $x \in X$. If $S: X \to X$ is a bounded linear operator such that $\|S\| < 1/2c(1+\sqrt{n})$, then $T+S \in \Phi_+(X)$, and* $\dim N(T+S) \leqslant n$.

Proof. Since $N := N(T)$ is a finite dimensional subspace of a Banach space X, there exists a closed subspace W of X such that $X = N \oplus W$. Put P_W be the projection on X to N along W. It is a bounded linear operator in general. The best possible estimate of its norm is known as follows. There exists a closed set Z such that $X = N \oplus Z$ and $\|P_Z\| \leqslant \sqrt{n}$, see [14, B.4.9, pp. 29–30, 28.2.6, p. 386]. Set $(I - P_Z)\,x = z \in Z$ for $x \in X$. Suppose that $y \in [z]$. Then $y = y - z + z$ implies that $y - z = P_Z(y)$. Hence

$$|z| \leqslant |z-y| + |y| \leqslant \|P_Z\|\,|y| + |y| \leqslant (1+\sqrt{n})\,|y|.$$

Since this inequality holds for all $y \in [z]$, we have that $|z| \leqslant (1+\sqrt{n})\,|[z]|$. Let $z = (I - P_Z)\,x$ for some $x \in X$. Since $P_Z x \in N$, we have that $[z] = [x]$; hence $|(I-P_Z)\,x| \leqslant (1+\sqrt{n})\,|[x]|$ for $x \in X$.

We apply this to our case. Take the space Z mentioned in the above for the space $N(T)$. From the condition in the theorem as well as this result, we obtain that $|(I-P_Z)\,x| \leqslant c(1+\sqrt{n})\,|Tx|$ for $x \in X$. Thus $c_Z = c(1+\sqrt{n})$, and we obtain the desired result by applying Lemma 7.2.

We obtain the following result by combining Theorem 1.2, Lemma 6.1 and Proposition 7.3. Denote by $\mathcal{S}_L(\Omega)$ the set of ω-periodic solutions for Eq. $(\mathrm{P}_\omega \mathrm{L})$.

PROPOSITION 7.4. *Suppose that $\mathcal{B}$ is a fading memory space with the axiom* (B-3), *$I - T(\omega) \in \Phi_+(E)$ and that there exists a positive constant c such that $|[\phi]| \leqslant c\,|(I - \hat{T}(\omega))\,\phi|_{\mathcal{B}}$ for $\phi \in \mathcal{B}$. Let $n =$* $\dim N(I - T(\omega))$. *If $\|K(\omega, 0)\| < 1/2c(1+\sqrt{n})$, and if Eq. $(P_\omega L)$ has an E-bounded solution, then $\mathcal{S}_L(\omega) \neq \varnothing$ and* $\dim \mathcal{S}_L(\omega) \leqslant n$.

In the proof of Theorem 6.5, we have the estimate for the constant c such that $|U\psi|_{\mathcal{B}} \leqslant c\,|\psi|_{\mathcal{B}}$ for $\psi \in R(I - \hat{T}(\omega))$ as

$$c \leqslant M_0/(1 - e^{-\varepsilon_0 \omega}) + \sup\{\|T(t)\| : 0 \leqslant t \leqslant \omega\}\ \|S_M^{-1}\|.$$

Let P_M be the projection on E to N along M. Since $I - T(\omega) \in \Phi_+(E)$, there exists a constant $\gamma_M > 0$ such that $|(I - P_M)\,x)| \leqslant \gamma_M |(I - T(\omega))\,x| = \gamma_M\,|(I - T(\omega))(I - P_M)\,x|$ for $x \in E$. Hence, we have $\|S_M^{-1}\| \leqslant \gamma_M$.

THEOREM 7.5. *Suppose that UC_g is a uniform fading memory space, $I - T(\omega) \in \Phi_+(E)$, and that $\|T(t)\| \leqslant M_w e^{wt}$ for $t \geqslant 0$. Let $n :=$ $\dim N(I - T(\omega))$, and γ_M the constant defined as in the above. Let M_0, ε_0 be the positive constants, determined by the space UC_g, such that $\|S_0(t)\| \leqslant M_0 e^{-\varepsilon_0 t}$ for $t \geqslant 0$. Set*

$$c = M_0/(1 - e^{-\varepsilon_0 \omega}) + \sup\{\|T(t)\| : 0 \leqslant t \leqslant \omega\}\ \gamma_M,$$

and $a = JM_w \|B\|_\infty$, where $J = \sup_{\theta \leqslant 0} 1/g(\theta)$. Suppose that $\|B\|_\infty$ satisfies the condition

$$a \int_0^\omega e^{(a + w_+)(\omega - s)} \|\hat{T}(s)\|\, ds < 1/2c(1 + \sqrt{n}),$$

and that Eq. $(P_\omega L)$ has an E-bounded solution. Then $\mathscr{S}_L(\omega) \neq \emptyset$ and $\dim \mathscr{S}_L(\omega) \leqslant n$.

Proof. This follows form Proposition 3.4, Theorem 6.6 and Proposition 7.4.

8. THE EXISTENCE AND UNIQUENESS OF PERIODIC SOLUTIONS

We consider the existence of bounded solutions, and the existence and uniqueness of ω-periodic solutions for Eq. $(\mathrm{P}_\omega \mathrm{L})$. For each $\phi \in \mathscr{B}$ take the space $BC(\phi)$ the set of bounded, continuous functions $x: [0, \infty) \to E$ such that $x(0) = \phi(0)$. This is a complete metric space with the metric $d(x, y) = \|x - y\|_\infty := \sup\{|x(t) - y(t)| : t \geqslant 0\}$. Define an operator F_ϕ on $BC(\phi)$ by

$$(F_\phi x)(t) = T(t)\,\phi(0) + \int_0^t T(t - s)(B(s, \tilde{x}_s) + F(s))\, ds, \qquad t \geqslant 0, \tag{8.1}$$

where $\tilde{x}(t) = \phi(t)$ for $t \leqslant 0$ and $\tilde{x}(t) = x(t)$ for $t \geqslant 0$. Set

$$\|F\|_\infty = \sup\{|F(t)| : t \geqslant 0\}.$$

PROPOSITION 8.1. *Suppose that $\mathscr{B}$ is a fading memory space, $\|B\|_\infty$ and $\|F\|_\infty$ are finite, and that $T(t)$ is a C_0-semigroup on E such that there exist M_w, $w > 0$ for which $\|T(t)\| \leqslant M_w e^{-wt}$ for $t \geqslant 0$. If $JM_w \|B\|_\infty < w$, then every solution of Eq. $(P_\omega L)$ is E-bounded.*

Proof. It suffices to show that F_ϕ is a contraction for each $\phi \in \mathscr{B}$. It is obvious that $(F_\phi x)(t)$ is continuous for $t \geqslant 0$. Since $|\tilde{x}_s|_{\mathscr{B}} \leqslant J \|x\|_\infty + M |\phi|_{\mathscr{B}}$, $x \in BC(\phi)$, and since

$$|F_\phi x(t)| \leqslant M_w e^{-wt} |\phi(0)| + \int_0^t M_w e^{-w(t-s)}$$

$$\times [\|B\|_\infty (J\|x\|_\infty + M |\phi|_{\mathscr{B}}) + \|F\|_\infty] \, ds,$$

we have $\|F_\phi x\|_\infty \leqslant M_w |\phi(0)| + M_w w^{-1} [\|B\|_\infty (J\|x\|_\infty + M |\phi|_{\mathscr{B}}) + \|F\|_\infty]$; that is, $F_\phi x \in BC(\phi)$. In the similar manner, we have also that $\|F_\phi x - F_\phi y\|_\infty \leqslant w^{-1} J M_w \|B\|_\infty \|x-y\|_\infty$. Therefore, if $JM_w \|B\|_\infty < w$, then F_ϕ is a contraction on $BC(\phi)$, and has a unique fixed point, z, in $BC(\phi)$. Then $\tilde{z}$ is an E-bounded solution of Eq. $(\mathrm{P}_\omega \mathrm{L})$.

THEOREM 8.2. *Suppose that $\mathscr{B}$ is a uniform fading memory space satisfying the additional axiom* (B-3) *and that $\|T(t)\| \leqslant M_w e^{-wt}$ for $t \geqslant 0$. Let $c > 0$ be the constant such that $|\phi|_{\mathscr{B}} \leqslant c \, |(I - \hat{T}(\omega)) \, \phi|_{\mathscr{B}}$ for all $\phi \in \mathscr{B}$. If $\|B\|_\infty$ satisfies the condition*

$$JM_w (1 + 2c(JM_w H + M)) \, \|B\|_\infty < w,$$

then Eq. $(\mathrm{P}_\omega \mathrm{L})$ *has a unique ω-periodic solution v, and*

$$\|v\|_\infty \leqslant \frac{MM_w}{w - JM_w \|B\|_\infty} \|F\|_\infty.$$

Proof. The existence of the constant c follows from Theorem 4.8 (1). Let $\|B\|_\infty$ satisfy the condition in Theorem 8.2. Then, since $w - JM_w \|B\|_\infty$ becomes positive, Eq. $(\mathrm{P}_\omega \mathrm{L})$ has E-bounded solutions from Proposition 8.1. To show the existence of ω-periodic solutions of Eq. $(\mathrm{P}_\omega \mathrm{L})$, we will estimate $\|K(\omega, 0)\|$. From Proposition 3.7 and the condition in this theorem, we have

$$\|K(\omega, 0)\| \leqslant \frac{JM_w \|B\|_\infty (JM_w H + M)}{w - JM_w \|B\|_\infty} < \frac{1}{2c}.$$

From Proposition 7.4 it follows that $\mathscr{S}_L(\omega)$ is nonempty and $\dim \mathscr{S}_L(\omega) \leqslant \dim N(I - T(\omega)) = 0$ because $\tau^\nu < 0$. Therefore, Eq. $(\mathrm{P}_\omega \mathrm{L})$ has a unique ω-periodic solution, $v(t)$.

Set $\psi = v_0$. Then v is the fixed point of F_ψ defined by (8.1). Since $\|v\|_\infty := \sup\{|v(t)| : t \geqslant 0\} = \sup\{|v(t)| : -\infty < t < \infty\}$, it follows from Lemma 2.1 that $|v_s|_{\mathscr{B}} \leqslant J \|v\|_\infty$ for $s \geqslant 0$. Then we have that $|B(s, v_s)| \leqslant J \|B\|_\infty \|v\|_\infty$, and that $|v(t)| = |F_\psi v(t)| \leqslant M_w e^{-wt} |\psi(0)| + M_w w^{-1}(J \|B\|_\infty \|v\|_\infty + M \|F\|_\infty)$ for $t \geqslant 0$. Hence, for $t \geqslant 0$ and for $n = 1, 2, ...,$

$$|v(t)| = |v(t+n\omega)| \leqslant M_w e^{-w(t+n\omega)} |\psi(0)| + M_w w^{-1}(J \|B\|_\infty \|v\|_\infty + M \|F\|_\infty).$$

Taking the limit as $n \to \infty$, we have that

$$|v(t)| \leqslant M_w w^{-1}(J \|B\|_\infty \|v\|_\infty + M \|F\|_\infty)$$

for $t \geqslant 0$, which implies that $\|v\|_\infty \leqslant M_w w^{-1}(J \|B\|_\infty \|v\|_\infty + M \|F\|_\infty)$. Hence v satisfies the inequality in the theorem and the proof is complete.

COROLLARY 8.3. *Let $\mathscr{B} = UC_{-\gamma}$, $\gamma > 0$, and $\|T(t)\| \leqslant e^{-wt}$, $w > 0$. If*

$$\left(1 + \frac{12}{1 - e^{-\min\{\gamma, w\}\,\omega}}\right) \|B\|_\infty < w,$$

then Eq. $(P_\omega L)$ has a unique ω-periodic solution v, and

$$\|v\|_\infty \leqslant \frac{2}{w - \|B\|_\infty} \|F\|_\infty.$$

Proof. It is easy to see that $J = 1$, $H = 1$, $M_w = 1$, $\|S_0(t)\| = e^{-\gamma t}$ and $M = 2$ in Theorem 8.2. Now we will compute the value of the constant c in Theorem 7.4. From the assumption we have $\|T(\omega)\| < 1$, and hence,

$$\|S_M^{-1}\| = \|(I - T(\omega))^{-1}\| \leqslant 1/(1 - \|T(\omega)\|) \leqslant 1/(1 - e^{-w\omega}),$$

Thus we get

$$c \leqslant 1/(1 - e^{-\gamma\omega}) + 1/(1 - e^{-w\omega}) \leqslant 2/(1 - e^{-\min\{\gamma, w\}\,\omega}).$$

Therefore it follows from the assumption that all conditions of Theorem 8.2 are satisfied, and the proof is complete.

COROLLARY 8.4. *Let $\mathscr{B} = UC_{-\gamma}$, $\gamma > 0$, and $\|T(t)\| \leqslant M_w e^{-wt}$, $w > 0$, $M_w \geqslant 1$. If $w\omega > \log M_w$ and*

$$M_w \left(1 + \frac{2(M_w + 1)(M_w + 2)}{1 - \max\{e^{-\gamma\omega}, M_w e^{-w\omega}\}}\right) \|B\|_\infty < w,$$

then Eq. $(\mathrm{P}_\omega \mathrm{L})$ *has a unique ω-periodic solution v, and*

$$\|v\|_\infty \leqslant \frac{2M_w}{w - \|B\|_\infty M_w} \|F\|_\infty.$$

Now, we shall see, by means of a simple example, how the result of Corollary 8.3 can be used to prove existence of a periodic solution of a partial differential-integral equation.

EXAMPLE. Denote by $E = C[-\infty, \infty]$, the space of all continuous real valued functions $u(x)$, defined on $(-\infty, \infty)$, satisfying the condition that $\lim_{x\to-\infty} u(x)$ and $\lim_{x\to+\infty} u(x)$ exist, and take its norm as $\|u\| = \sup_{-\infty<x<+\infty} |u(x)|$. Then E is a Banach space.

We consider the initial value problem for the equation of the form

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - \alpha u(t,x) + b(t,x) \int_{-\infty}^{t} e^{-c(t-s)} u(s,x)\, ds + f(t,x), \tag{8.2}$$

$$u(\theta, x) = \phi(\theta, x), \qquad \theta \leqslant 0, \quad \phi \in \mathscr{B},$$

where $\mathscr{B} = UC_{-\gamma}$.

It is well known that the linear operators A and A_0, defined by

$$Au = \frac{d^2u}{dx^2} - \alpha u \quad \text{for} \quad u \in \mathscr{D}(A), \qquad A_0 u = \frac{d^2u}{dx^2} \quad \text{for} \quad u \in \mathscr{D}(A_0)$$

and

$$\mathscr{D}(A) = \mathscr{D}(A_0) = \left\{ u \in E : \frac{d}{dx} u, \frac{d^2}{dx^2} u \in E \right\},$$

are infinitesimal generators of C_0-semigroups $T(t)$ and $T_0(t)$ on E, respectively, cf. [3, Chapter VIII]. Then

$$\|T_0(t)\| = 1,\ T(t) = e^{-\alpha t} T_0(t), \qquad \text{and} \qquad \|T(t)\| = e^{-\alpha t}$$

for all $t \geqslant 0$.

Assume that

(C-1) $\alpha > 0$ and $c > \gamma > 0$.

(C-2) $b(t, x)$ and $f(t, x)$: $R \times R \to R$ are continuous and ω-periodic functions in t such that $b(t, \cdot), f(t, \cdot) \in E$, $t \in R$.

Put $\|b(t)\| = \sup_{-\infty < x < \infty} |b(t, x)|$, $\|b\|_\infty = \sup_{0 \leqslant t \leqslant \omega} \|b(t)\|$. Similarly, we define $\|f(t)\|$ and $\|f\|_\infty$ for $f(t, x)$. Set

$$B(t, \phi)(x) = b(t, x) \int_{-\infty}^{0} e^{c\theta} \phi(\theta, x)\, d\theta, \phi \in \mathscr{B}.$$

Then we have

$$|B(t, \phi)(x)| \leqslant \|b(t)\| \int_{-\infty}^{0} e^{(c-\gamma)\theta}\, |\phi|_{\mathscr{B}}\, d\theta = \frac{\|b(t)\|}{c-\gamma}\, |\phi|_{\mathscr{B}}$$

and hence, $\|B\|_\infty \leqslant \|b\|_\infty/(c-\gamma)$. Therefore from Corollary 8.3 we have the following result.

THEOREM 8.5. *Assume that the conditions* (C-1) *and* (C-2) *are satisfied. If*

$$\left(1 + \frac{12}{1 - \exp(-\min\{\alpha, \gamma\}\, \omega)}\right) \frac{\|b\|_\infty}{c-\gamma} < \alpha,$$

then Eq. (8.2) *has a unique ω-periodic solution v, and*

$$\|v\|_\infty \leqslant \frac{2}{\alpha - \|b\|_\infty/(c-\gamma)}\, \|f\|_\infty.$$

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